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Boundary eigensolutions in elasticity. I. Theoretical development

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Abstract

The theory of fundamental boundary eigensolutions for elastostatic boundary value problems is developed. The underlying fundamental eigenproblem is formed by inserting the eigenparameter and a tensor weight function into the boundary condition, rather than into the governing differential equation as is often done for vibration problems. The resulting spectra are real and the eigenfunctions (eigendeformations) are mutually orthogonal on the boundary, thus providing a basis for solutions. The weight function permits effective treatment of non-smooth problems associated with cracks, notches and mixed boundary conditions. Several ideas related to the behavior of eigensolutions in the domain, integral equation methods, variational methods, convergence characteristics, flexibility and stiffness kernels, and solutions to problems with body forces are also introduced. Of particular note are the integral equation and variational formulations that lead to the development of new computational formulations for boundary element and finite element methods, respectively. An example with closed form and numerical results is included to illustrate some aspects of the theory.

Keywords: Theory of elasticity; Boundary value problems; Non-smooth problems; Eigensolutions; Generalized Fourier analysis; Computational mechanics

1. Introduction

The general theory of fundamental boundary eigensolutions and its relation with the direct integral equation formulation has been presented for the potential problem in Hadjesfandiari (1998) and Hadjesfandiari and Dargush (2001a). The application of this theory in computational methods has also been developed for the potential problem in Hadjesfandiari and Dargush (2001b). The finite element and boundary element formulations presented there are completely consistent with the theory of potential boundary value problems, including all those problems that are classified as non-smooth. By adopting this approach, we obtain a clearer understanding of the computational methods, along with general numerical algorithms for the practical solution of non-smooth problems. This theory may be generalized to every

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boundary value problem. The field function can be scalar, vector or tensor. In this paper, we develop the theory further within the context of elastostatic boundary value problems.

Based upon the integral equation representation of an elastostatic problem, the boundary surface must play a key role in the solution. Consequently, it would seem appropriate to employ basis deformations that are orthogonal over the boundary. We will see that these deformations can be generated by solving an eigenproblem in which the eigenparameter appears in the boundary condition. The new concepts that emerge from this approach seem to have significance for the general theory of elasticity as well as for computational mechanics.

An elastostatic boundary value problem is considered non-smooth if the domain is non-smooth or mixed boundary conditions are specified. In these cases, the solution is non-analytic at some points on the boundary. Thus, the characteristic feature of these non-smooth problems is the presence of singularities in the traction or higher order derivatives on the boundary. The power of these singularities may be obtained by a local analysis with homogeneous boundary conditions following, for example, the approach of Williams (1952). Since most of the problems posed in engineering applications involve either non-smooth geometries or mixed boundary conditions, we attempt to provide a unified treatment that encompasses non-smooth problems.

To some readers, the term fundamental eigendeformation might seem to be related to the eigenstrain term used by Eshelby (1957) to refer to stress-free transformation strains. We emphasize that the fundamental boundary deformations presented here are a completely different concept and are also different from the familiar dynamic eigenmodes used in vibration theory. The new theory of boundary eigensolutions will be introduced in Section 3. First, however, some well-known relations from elasticity are presented in the following section for later reference.

2. Basic equations in elasticity

Equilibrium of the stress state σ_{ij} in the absence of body forces and moments leads to the following equations (e.g., Kupradze, 1979)

$$\sigma_{ij,j} = 0 \tag{2.1}$$

$$\sigma_{ij} = \sigma_{ji} \tag{2.2}$$

in domain V, subject to boundary conditions on S. The domain V can be two or three dimensional, simply or multiply connected. The boundary S is a contour or set of contours in two dimensional (2-D) problems and a closed surface or surfaces for three dimensional (3-D) domains.

The strain tensor ε_{ij} is defined in terms of displacements u_i (for small deformations only) as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3}$$

We consider the material to be homogeneous and linear elastic where the relation between the stress tensor σ_{ij} and strain tensor ε_{ij} is

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijkl}u_{k,l} \tag{2.4}$$

with C_{ijkl} as the tensor of elastic constants which has the following symmetry relations

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{2.5}$$

By substituting the stress tensor from Eq. (2.4) in the equilibrium equation, we obtain the Navier equations expressing equilibrium in terms of displacements as

$$C_{ijkl}u_{k,lj} = 0 \tag{2.6}$$

The traction vector on the boundary is

$$t_i = \sigma_{ij} n_j \tag{2.7a}$$

or in terms of deformation

$$t_i = C_{ijkl}\varepsilon_{kl} n_j = C_{ijkl} u_{k,l} n_j \tag{2.7b}$$

with n_j representing the unit outward normal vector.

In general three different boundary conditions may be specified. These are: Dirichlet ($\boldsymbol{u} = \bar{\boldsymbol{u}}$ on S), Neumann ($\boldsymbol{t} = \bar{\boldsymbol{t}}$ on S) and mixed ($\boldsymbol{u} = \bar{\boldsymbol{u}}$ on S_u and $\boldsymbol{t} = \bar{\boldsymbol{t}}$ on S_t with $S_u \cup S_t = S$ and $S_u \cap S_t = \emptyset$). If the problem is well posed, the Dirichlet and mixed problems have unique solutions. The solution for the Neumann problem is unique excluding an arbitrary rigid body motion.

As we know from the theory of boundary value problems, $\boldsymbol{u}$ is analytic in the domain V. However, displacement $\boldsymbol{u}$ need not be analytic on the boundary S. Furthermore, at non-smooth points the traction $\boldsymbol{t}$ is not defined. In general, $\boldsymbol{t}$ is a piecewise continuous function on the boundary. It should be noted that although the stress tensor and traction can be singular at non-smooth points, the criteria for an acceptable singularity is the existence of a bounded strain energy in the domain.

Let σ^u_{ij} and σ^v_{ij} be stresses corresponding to deformations u_i and v_i which satisfy the equilibrium equation. From the reciprocal theorem, we have

$$\int_S t^u_i v_i \, \mathrm{d}S = \int_S t^v_i u_i \, \mathrm{d}S \tag{2.8}$$

We also have for every acceptable deformation

$$\int_S t^u_i u_i \, \mathrm{d}S = \int_V \sigma^u_{ij}\varepsilon^u_{ij} \, \mathrm{d}V \tag{2.9}$$

which means that the work of external forces is twice the strain energy $\mathscr{U}$, where

$$\mathscr{U} = \frac{1}{2}\int_V \sigma^u_{ij}\varepsilon^u_{ij} \, \mathrm{d}V$$

By definition the total energy is

$$\Pi = \mathscr{U} + \mathscr{V} \tag{2.10}$$

where $\mathscr{V}$ is the potential of the external forces

$$\mathscr{V} = -\int_S t_i u_i \, \mathrm{d}S \tag{2.11}$$

Then from Eq. (2.9)

$$\Pi = -\mathscr{U} \tag{2.12}$$

which actually is the minimum for equilibrium.

3. Fundamental boundary eigensolutions

3.1. Introduction

In this section we develop the theory of fundamental boundary eigenexpansion in elastostatics. This theory gives us a powerful tool to analyze every problem, whether it is solved in closed form or approximately. In fact, the beauty of this approach is its ability to account for singularity in the solution of boundary value problems in elasticity. Furthermore, the energy concepts developed in this section show the profound nature of the theory of eigenexpansion.

3.2. The boundary eigenproblem

The *fundamental boundary eigenproblem* for elastostatics is defined as follows: find non-zero deformation $\boldsymbol{u}$ such that in the domain V

$$\sigma_{ij,j} = C_{ijkl}u_{k,lj} = 0 \tag{3.1a}$$

and on the boundary S

$$t_i = \lambda\varphi_{ij}u_j \tag{3.1b}$$

where λ is an eigenvalue, and φ_{ij} is a symmetric positive definite tensor which is piecewise continuous and integrable on the boundary. This tensorial function is called the *tensor weight function*. In practical problems we can usually choose this tensor as

$$\varphi_{ij} = \varphi\delta_{ij}$$

where δ_{ij} is the Kronecker delta and φ a positive piecewise continuous integrable scalar function. By doing so, the fundamental boundary condition reduces to

$$t_i = \lambda\varphi u_i$$

The deformation u_i is a vectorial eigenfunction which is called an *eigenmode* or *eigendeformation.*

With the classical approach, the eigenparameter is introduced into the governing differential equation and a specific set of homogeneous boundary conditions are prescribed. In the boundary eigenproblems (3.1a) and (3.1b) however, the elastostatic differential operator remains intact, while the eigenvalue is inserted into the boundary condition. Thus the eigendeformations u_i associated with Eqs. (3.1a) and (3.1b) satisfy equilibrium in the domain V. Furthermore, the infinite sequence of eigenmodes for Eqs. (3.1a) and (3.1b) can be used as a basis for all solutions to boundary value problems in the domain V governed by the Navier equations with arbitrary well-posed boundary conditions on S.

3.3. Rigid body eigenmodes

We know that rigid body motion cannot generate any strain or stress. Consequently, a rigid body motion is an eigenmode corresponding to $\lambda = 0$ with the general form

$$u_i = (a_{ij} - \delta_{ij})x_j + c_i$$

where a_{ij} is a constant proper orthogonal tensor showing rotation and c_i is a constant vector showing translation. In 2-D the multiplicity of $\lambda = 0$ is 3 (two translations and one rotation) and in 3-D the multiplicity is 6 (three translations and three rotations). Of course, here in the small deformation theory of elasticity, we deal with infinitesimal values for rotation and translation. We can show in the limit that the tensor $(a_{ij} - \delta_{ij})$ approaches to an anti-symmetric tensor. Rigid body motion can then be written in terms of a rotation vector Ω_i. Using the cross product

$$\boldsymbol{u} = \underset{\sim}{\Omega} \times \boldsymbol{x} + \boldsymbol{c}$$

or in index form we have

$$u_i = \varepsilon_{ijk}\Omega_j x_k + c_i$$

where ε_{ijk} is the permutation tensor.

3.4. Rayleigh quotient and eigenenergy

Multiplying both sides of the fundamental boundary condition (3.1b) with u_i and integrating on the boundary, we obtain

$$\int_S t_i u_i \, \mathrm{d}S = \lambda \int_S \varphi_{ij} u_i u_j \, \mathrm{d}S$$

Then

$$\lambda = \frac{\int_S t_i u_i \, \mathrm{d}S}{\int_S \varphi_{ij} u_i u_j \, \mathrm{d}S} \tag{3.2}$$

We also know that

$$\int_S t_i u_i \, \mathrm{d}S = \int_V \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}V$$

Therefore

$$\lambda = \frac{\int_V \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}V}{\int_S \varphi_{ij} u_i u_j \, \mathrm{d}S} = \frac{\int_V C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, \mathrm{d}V}{\int_S \varphi_{ij} u_i u_j \, \mathrm{d}S} \tag{3.3a}$$

where the right-hand side of this relation is the *Rayleigh quotient*. In general we write

$$R\{\boldsymbol{u}\} = \frac{\int_V \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}V}{\int_S \varphi_{ij} u_i u_j \, \mathrm{d}S} \tag{3.3b}$$

We already know that the Rayleigh quotient for a fundamental eigenfunction is the corresponding fundamental eigenvalue. The numerator of this quotient is twice the strain energy of the body. If we normalize u_i, such that

$$\int_S \varphi_{ij} u_i u_j \, \mathrm{d}S = 1 \tag{3.4}$$

then for boundary eigenfunctions

$$\lambda = \int_V \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}V \tag{3.5}$$

We may call λ the *eigenenergy with respect to the weight function* φ_{ij}. In other words, the strain energy generated by an eigenmode is half of the corresponding eigenenergy. That is,

$$\mathscr{U} = \tfrac{1}{2}\lambda \tag{3.6}$$

3.5. Basic properties of the boundary eigensolutions

As in the potential problem, we have the following theorems for the fundamental eigenproblem of elastostatics defined by Eqs. (3.1a) and (3.1b):

Theorem 1. *At least one eigensolution exists.*

Theorem 2. *All of the eigenvalues are real.*

Proof. Let (λ, u_i) represent an eigensolution of Eqs. (3.1a) and (3.1b). If (λ, u_i) is complex, its complex conjugate is also an eigensolution of Eqs. (3.1a) and (3.1b). By using the reciprocal theorem for u_i and $\bar{u}_i$, we obtain

$$\int_S t_i \bar{u}_i \, dS = \int_S \bar{t}_i u_i \, dS$$

Using the fundamental boundary condition, this becomes

$$\lambda \int_S \varphi_{ij} u_j \bar{u}_i \, dS = \bar{\lambda} \int_S \varphi_{ij} \bar{u}_j u_i \, dS$$

By interchanging indices i and j on the right-hand side and putting $\varphi_{ij} = \varphi_{ji}$, we obtain

$$(\lambda - \bar{\lambda}) \int_S \varphi_{ij} u_i \bar{u}_j \, dS = 0$$

However φ_{ij} is positive definite everywhere on the boundary, thus

$$\int_S \varphi_{ij} u_i \bar{u}_j \, dS > 0$$

This requires $\lambda = \bar{\lambda}$, which means that λ is a real number. □

Theorem 3. *All non-zero eigenvalues are positive.*

Proof. This follows directly from the Rayleigh quotient expressed in Eqs. (3.3a) and (3.3b). □

Theorem 4. *The sequence of eigenmodes are boundary orthonormal with respect to* φ_{ij}.

Proof. We prove this theorem for distinct eigenvalues. Let $(\lambda_1, u_i^{(1)})$ and $(\lambda_2, u_i^{(2)})$ be two eigenmodes where $\lambda_1 \neq \lambda_2$. By applying the reciprocal theorem for these two eigenmodes we obtain

$$\int_S t_i^{(1)} u_i^{(2)} \, dS = \int_S t_i^{(2)} u_i^{(1)} \, dS$$

Using the fundamental boundary condition, this becomes

$$\lambda_1 \int_S \varphi_{ij} u_j^{(1)} u_i^{(2)} \, dS = \lambda_2 \int_S \varphi_{ij} u_j^{(2)} u_i^{(1)} \, dS$$

By interchanging indices i and j on the right-hand side and putting $\varphi_{ij} = \varphi_{ji}$, we obtain

$$(\lambda_1 - \lambda_2) \int_S \varphi_{ij} u_i^{(1)} u_j^{(2)} \, dS = 0$$

However since $\lambda_1 \neq \lambda_2$, we have

$$\int_S \varphi_{ij} u_i^{(1)} u_j^{(2)} \, dS = 0 \tag{3.7}$$

Gram–Schmidt orthogonalization can be used for eigenmodes associated with non-distinct eigenvalues. In Appendix A the normalized rigid body eigenmodes corresponding to $\lambda = 0$ are given for 3-D and 2-D cases when $\varphi_{ij} = \varphi \delta_{ij}$. □

In general, we have

$$\int_S \varphi_{ij} u_i^{(m)} u_j^{(n)} \, \mathrm{d}S = 0, \quad m \neq n \tag{3.8a}$$

and by assuming normalized eigensolutions

$$\int_S \varphi_{ij} u_i^{(m)} u_j^{(m)} \, \mathrm{d}S = 1, \quad m = 1, 2, \ldots \tag{3.8b}$$

Now, we investigate the properties of fundamental eigenmodes in the domain. By multiplying Eq. (3.8a) with λ_m and using the fundamental boundary condition $t_j^{(m)} = \lambda_m \varphi_{ji} u_i^{(m)}$ with $\varphi_{ji} = \varphi_{ij}$, we obtain

$$\int_S t_j^{(m)} u_j^{(n)} \, \mathrm{d}S = 0, \quad m \neq n \tag{3.9a}$$

Similarly

$$\int_S t_i^{(n)} u_i^{(m)} \, \mathrm{d}S = 0, \quad m \neq n \tag{3.9b}$$

By adding Eqs. (3.9a) and (3.9b)

$$\int_S [t_i^{(m)} u_i^{(n)} + t_i^{(n)} u_i^{(m)}] \, \mathrm{d}S = 0$$

and using Eq. (2.7a) for the traction

$$\int_S [\sigma_{ij}^{(m)} u_i^{(n)} + \sigma_{ij}^{(n)} u_i^{(m)}] n_j \, \mathrm{d}S = 0$$

Now by applying the divergence theorem

$$\int_V [\sigma_{ij}^{(m)} u_i^{(n)} + \sigma_{ij}^{(n)} u_i^{(m)}]_{,j} \, \mathrm{d}V = 0$$

or

$$\int_V [\sigma_{ij,j}^{(m)} u_i^{(n)} + \sigma_{ij,j}^{(n)} u_i^{(m)} + \sigma_{ij}^{(m)} u_{i,j}^{(n)} + \sigma_{ij}^{(n)} u_{i,j}^{(m)}] \, \mathrm{d}V = 0 \tag{3.10}$$

From the equilibrium equation (3.1a)

$$\sigma_{ij,j}^{(m)} = 0, \quad \sigma_{ij,j}^{(n)} = 0$$

Then in Eq. (3.10), we have

$$\int_V [\sigma_{ij}^{(m)} \varepsilon_{ij}^{(n)} + \sigma_{ij}^{(n)} \varepsilon_{ij}^{(m)}] \, \mathrm{d}V = 0, \quad m \neq n$$

and by using Eq. (2.4) along with the symmetry conditions in Eq. (2.5)

$$\sigma_{ij}^{(m)} \varepsilon_{ij}^{(n)} = \sigma_{ij}^{(n)} \varepsilon_{ij}^{(m)}$$

Therefore

$$\int_V \sigma_{ij}^{(m)} \varepsilon_{ij}^{(n)} \, \mathrm{d}V = 0, \quad m \neq n \tag{3.11}$$

Furthermore, it is easily proved

$$\int_V C_{ijkl} u_{k,l}^{(m)} u_{i,j}^{(m)} \, dV = \lambda_m, \quad m = 1, 2, \ldots$$

or

$$\mathscr{U}_m = \frac{1}{2} \int_V \sigma_{ij}^{(m)} \varepsilon_{ij}^{(m)} \, dV = \frac{1}{2} \lambda_m \tag{3.12}$$

which is exactly Eq. (3.6).

Eqs. (3.11) and (3.12) represent properties of the eigenmodes in the domain V. These properties are most interesting from an energy perspective, which reveals that the eigenmodes are energy orthogonal. Notice that although the tensorial weight function φ_{ij} does not appear in the volume integrals, it does affect the eigenmodes.

3.6. Integral equation method

As is well known, every boundary value problem can be transformed into an integral equation. The direct boundary integral equation for the elastostatic problem without body force is written (e.g., Banerjee (1994))

$$c_{ij}(\xi) u_j(\xi) + \int_S F_{ij}(\xi, x) u_j(x) \, dS(x) = \int_S G_{ij}(\xi, x) t_j(x) \, dS(x) \tag{3.13}$$

where the kernel $G_{ij}(\xi, x)$ is the displacement in the i direction at point ξ generated by a concentrated unit force at x acting in the j direction. Thus, for the isotropic case

$$G_{ij}(\xi, x) = \begin{cases} \frac{1}{8\pi\tilde{\mu}(1-v)} \left[(3-4v)\delta_{ij} \ln \frac{1}{r} + \frac{y_i y_j}{r^2} \right] & \text{in 2-D} \\ \frac{1}{16\pi\tilde{\mu}(1-v)r} \left[(3-4v)\delta_{ij} + \frac{y_i y_j}{r^2} \right] & \text{in 3-D} \end{cases} \tag{3.14a}$$

$$F_{ij}(\xi, x) = \begin{cases} \frac{1}{4\pi(1-v)r} \left[(1-2v)\left(n_i \frac{y_j}{r} - n_j \frac{y_i}{r} + \delta_{ij} n_k \frac{y_k}{r} \right) + \frac{2 y_i y_j y_k}{r^3} n_k \right] & \text{in 2-D} \\ \frac{1}{8\pi(1-v)r^2} \left[(1-2v)\left(n_i \frac{y_j}{r} - n_j \frac{y_i}{r} + \delta_{ij} n_k \frac{y_k}{r} \right) + \frac{3 y_i y_j y_k}{r^3} n_k \right] & \text{in 3-D} \end{cases} \tag{3.14b}$$

where $y_i = x_i - \xi_i$, r is the distance between points x and ξ, and n_i is the outward unit vector normal at x on the surface. Meanwhile, the constant $\tilde{\mu}$ is the shear modulus and v is the Poisson ratio. It should be mentioned that the 2-D kernels in Eqs. (3.14a) and (3.14b) are for the plane strain case. The kernel corresponding to the plane stress case can be obtained from the plane strain kernel given above by using an effective Poisson ratio $\bar{v} = v/(1+v)$. By substituting the fundamental boundary condition Eq. (3.1b) into Eq. (3.13), we obtain the boundary eigenproblem in integral form

$$c_{ij}(\xi) u_j(\xi) + \int_S F_{ij}(\xi, x) u_j(x) \, dS(x) = \lambda \int_S G_{ij}(\xi, x) \varphi_{jk}(x) u_k(x) \, dS(x) \tag{3.15}$$

This is an integral representation of the fundamental eigenproblem (3.1a) and (3.1b). The solution of Eq. (3.15) has all of the characteristics defined previously. The eigensolutions of Eq. (3.15) are real, with non-negative eigenvalues and boundary orthogonal eigenmodes. Consequently, the spectrum of the direct integral equation representation of the elastostatic problem is real for every positive definite, integrable boundary weight function φ_{ij}, and the eigenmodes form an orthogonal set. Although the spectrum analysis of the indirect integral equation formulation has been studied extensively (Kupradze, 1979), a similar treatment has not appeared before in the literature for the direct method.

Furthermore, we can introduce the *weighted traction* $\boldsymbol{t}^{\varphi}$, where

$$t_i = \varphi_{ij} t_j^{\varphi} \tag{3.16}$$

Then Eq. (3.13) can be rewritten

$$c_{ij}(\xi)u_j(\xi) + \int_S F_{ij}(\xi, x)u_j(x)\,\mathrm{d}S(x) = \int_S G_{ij}(\xi, x)\varphi_{jk}(x)t_k^{\varphi}(x)\,\mathrm{d}S(x) \tag{3.17}$$

This equation will be further substantiated in the following sections.

In non-smooth problems involving stress and traction singularities, the weight function φ_{ij} can be chosen to capture the asymptotic behavior of the traction near the singular point. The suitable φ_{ij} may be derived from a local analysis (Williams, 1952). The integral equation (3.17) then involves only bounded solution variables $\boldsymbol{u}(x)$ and $\boldsymbol{t}^{\varphi}(x)$. In other words, $\boldsymbol{t}^{\varphi}(x)$ is piecewise regular.

In a practical sense for engineering application, we may wish to solve discretized versions of Eqs. (3.15) and (3.17) by using the boundary element method (e.g., Banerjee (1994)). Numerical solution of Eq. (3.15) allows us to study the character of the discretized integral equation representation of the elastostatic problem, while the computational algorithms associated with Eq. (3.17) permit the direct solution of boundary value problems. These new boundary element formulations will be presented in Part II of this paper (Hadjesfandiari and Dargush, 2001c), and then used to solve a series of non-smooth boundary value problems involving cracks, notches and bimaterial interfaces.

3.7. Variational formulation

From the Rayleigh quotient (3.3b), we can see that for any boundary eigensolution, say $(\lambda_n, \boldsymbol{u}_n)$, the functional $R\{\boldsymbol{u}_n\} = \lambda_n$. Furthermore, it is easy to show that the Rayleigh quotient is an extremum for boundary eigenmodes. Taking the first variation of $R\{\boldsymbol{u}\}$ from Eq. (3.3b), we obtain

$$\delta R\{\boldsymbol{u}\} = \frac{2\left[\int_V C_{ijkl}\varepsilon_{ij}\,\delta\varepsilon_{kl}\,\mathrm{d}V\right]\left[\int_S \varphi_{ij}u_i u_j\,\mathrm{d}S\right] - 2\left[\int_V C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\,\mathrm{d}V\right]\left[\int_S \varphi_{ij}u_i\,\delta u_j\,\mathrm{d}S\right]}{\left[\int_S \varphi_{ij}u_i u_j\,\mathrm{d}S\right]^2}$$

Substituting Eq. (3.3b) again produces

$$\delta R\{\boldsymbol{u}\} = 2\frac{\left[\int_V C_{ijkl}\varepsilon_{ij}\,\delta\varepsilon_{kl}\,\mathrm{d}V\right] - R\{\boldsymbol{u}\}\left[\int_S \varphi_{ij}u_i\,\delta u_j\,\mathrm{d}S\right]}{\left[\int_S \varphi_{ij}u_i u_j\,\mathrm{d}S\right]}$$

By using the divergence theorem, we obtain

$$\delta R\{\boldsymbol{u}\} = 2\frac{\int_S (t_i - R\{\boldsymbol{u}\}\varphi_{ij}u_j)\,\delta u_i\,\mathrm{d}S - \int_V \sigma_{ij,j}\,\delta u_i\,\mathrm{d}V}{\int_S \varphi_{ij}u_i u_j\,\mathrm{d}S}$$

Now δu_i is an arbitrary variation in the domain and on the boundary. For an extremum $R\{\boldsymbol{u}\}$,

$$\delta R\{\boldsymbol{u}\} = 0 \tag{3.18}$$

and we must have

$$\sigma_{ij,j} = 0 \quad \text{in } V$$

and

$$t_i = R\{\boldsymbol{u}\}\varphi_{ij}u_j \quad \text{on } S$$

This defines the fundamental boundary eigenproblem, where $R\{\boldsymbol{u}\} = \lambda$. Therefore, every eigenmode extremizes the Rayleigh quotient, and value of this quotient is the eigenvalue corresponding to the eigenmode.

As we said when the eigenmodes are orthonormal, twice the strain energy of the body for each eigenmode is equal to the corresponding eigenvalue. We have the following interpretation by using strain energy.

Extremize the strain energy

$$\mathscr{U} = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V \tag{3.19a}$$

such that

$$\int_S \varphi_{ij}u_iu_j\,\mathrm{d}S = \|\boldsymbol{u}\|^2 = 1 \tag{3.19b}$$

Define a new functional $\bar{\mathscr{U}}\{\boldsymbol{u}\}$ by using the Lagrange multiplier α such that

$$\bar{\mathscr{U}}\{\boldsymbol{u}\} = \mathscr{U}\{\boldsymbol{u}\} - \alpha\int_S \varphi_{ij}u_iu_j\,\mathrm{d}S \tag{3.20}$$

Taking the first variation of $\bar{\mathscr{U}}\{\boldsymbol{u}\}$ produces

$$\delta\bar{\mathscr{U}} = \int_V C_{ijkl}\varepsilon_{kl}\,\delta\varepsilon_{ij}\,\mathrm{d}V - 2\alpha\int_S \varphi_{ij}u_i\,\delta u_j\,\mathrm{d}S$$

Then after employing the divergence theorem, this becomes

$$\delta\bar{\mathscr{U}} = \int_S (t_i - 2\alpha\varphi_{ij})\,\delta u_i\,\mathrm{d}S - \int_V \sigma_{ij,j}\,\delta u_i\,\mathrm{d}S$$

Consequently, in order to have $\delta\bar{\mathscr{U}} = 0$ for arbitrary δu_i in the domain and on the boundary, we must satisfy

$$\sigma_{ij,j} = 0 \quad \text{in } V$$

and

$$t_i = 2\alpha\varphi_{ij}u_j \quad \text{on } S$$

Therefore $\boldsymbol{u}$ is a generalized fundamental eigenfunction and $\alpha = \frac{1}{2}\lambda$. For these extremum conditions corresponding to an eigensolution $(\underline{\lambda}, \underline{\boldsymbol{u}})$, we have

$$\mathscr{U}\{\underline{\boldsymbol{u}}\} = \frac{1}{2}\int_V (\underline{\sigma}_{ij}\underline{\varepsilon}_{ij})\,\mathrm{d}V = \frac{1}{2}\int_S \underline{t}_i\underline{u}_i\,\mathrm{d}S$$

Then using the generalized fundamental boundary condition, along with the constraint (3.19b), we find that $\mathscr{U}\{\underline{\boldsymbol{u}}\} = \frac{1}{2}\underline{\lambda}$. Thus, the Lagrange multipliers α in $\bar{\mathscr{U}}\{\boldsymbol{u}\}$ are the expected eigenvalues of the fundamental boundary eigenproblem (3.1a) and (3.1b) and the extremum values of $\mathscr{U}\{\boldsymbol{u}\}$ are exactly half of these eigenvalues.

What is the consequence of the extremum conditions?

The admissible deformation u_i which minimizes $\mathscr{U}$ under the constraint (3.19b) is the eigenmode $u_i^{(1)}$ and the minimum value of $\mathscr{U}$ is half of the corresponding eigenvalue λ_1. If we impose not only the normalizing condition (3.19b), but also the orthogonality condition

$$\int_S \varphi_{ij}u_i^{(1)}u_j\,\mathrm{d}S = 0$$

then the function which minimizes $\mathscr{U}$ is eigenmode $u_i^{(2)}$ and the minimum value of the strain energy $\mathscr{U}\{\boldsymbol{u}^{(2)}\} = \frac{1}{2}\lambda_2$ is half of the associated eigenvalue λ_2. Continuing this process, the successive minimum problems

$$\text{minimize} \quad \mathscr{U}\{\boldsymbol{u}\} = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V$$

subject to $\int_S \varphi_{ij} u_i u_j \, \mathrm{d}S = 1$

and

$$\int_S \varphi_{ij} u_i^{(m)} u_j \, \mathrm{d}S = 0, \quad m = 1, 2, \ldots, n-1$$

define the eigenmodes $\boldsymbol{u}^{(n)}$ and half of the corresponding eigenvalue λ_n equals the minimum value $\mathscr{U}\{\boldsymbol{u}^{(n)}\}$.

We can deal with the Rayleigh quotient (3.3a) and (3.3b) instead of the strain energy $\mathscr{U}$. Then we drop the normalization condition and the minimization problem becomes:

$$\text{minimize} \quad R\{\boldsymbol{u}\} = \frac{\int_V \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}V}{\int_S \varphi_{ij} u_i u_j \, \mathrm{d}S}$$

$$\text{subject to} \quad \int_S \varphi_{ij} u_i^{(m)} u_j \, \mathrm{d}S = 0, \quad m = 1, 2, \ldots, n-1$$

which again define the eigenmodes $\boldsymbol{u}^{(n)}$ and the corresponding eigenvalue λ_n equals the minimum value $R\{\boldsymbol{u}^{(n)}\}$.

3.8. Completeness of the system of boundary eigensolutions

Based upon the results from the previous section, we have the following theorems:

Theorem 5. *The fundamental boundary eigenproblem has an infinite collection of eigenvalues such that*

(i) *the eigenvalues form an increasing sequence,* $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \ldots$,
(ii) *the eigenvalues* λ_n *become infinite for* $n \to \infty$.

Condition (ii) implies, in particular, that each eigenvalue has only a finite number of multiplicity, and that only a finite number of eigenvalues can be negative. From Theorem 3, of course we know there are no negative eigenvalues. The most important consequence of the unboundedness of the eigenvalues is the completeness of the system of eigenfunctions.

Theorem 6. *The system of eigenmodes of the fundamental problem is complete.*

Proof. Assume $\boldsymbol{w}$ to be an L_2-function on the boundary S with respect to Φ, which means

$$\int_S \varphi_{ij} w_i w_j \, \mathrm{d}S = \|\boldsymbol{w}\|^2 < \infty \tag{3.21}$$

We can imagine that $\boldsymbol{w}$ is the boundary value of a solution of an elasticity problem in the domain V. Approximate $\boldsymbol{w}$ by the first N eigenmodes and let

$$\boldsymbol{w}^N = \sum_{n=1}^{N} c_n \boldsymbol{u}^{(n)} \quad \text{in } V \cup S$$

Then by defining

$$\boldsymbol{e}^N = \boldsymbol{w} - \boldsymbol{w}^N$$

an approximation in the mean implies that the following error is a minimum:

$$E^N = \int_S \varphi_{ij} e_i^N e_j^N \, \mathrm{d}S \geqslant 0$$

Assume that the eigenmodes are normalized. Then it can be proved very easily that for minimum error

$$c_n = \int_S \boldsymbol{w} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{u}^{(n)} \, \mathrm{d}S = \int_S \varphi_{ij} w_i u_j^{(n)} \, \mathrm{d}S$$

By inserting these values, the minimum error is

$$E^N = \int_S \varphi_{ij} w_i w_j \, \mathrm{d}S - \sum_{n=1}^{N} c_n^2 \geqslant 0$$

which is the Bessel inequality. For completeness we have to prove that the equality holds in the limit as $N \to \infty$.

We know

$$\int_S \varphi_{ij} e_i^N u_j^{(n)} \, \mathrm{d}S = 0 \quad \text{for } n = 1, 2, \ldots, N$$

and because $\boldsymbol{w}$ is an elastic solution which satisfies equilibrium

$$\int_V \sigma_{ij}^{e^N} \varepsilon_{ij}^{(n)} \, \mathrm{d}V = 0 \quad \text{for } n = 1, 2, \ldots, N$$

From the variational method and the minimal property of λ_{N+1}, we have

$$\lambda_{N+1} \int_S \varphi_{ij} e_i^N e_j^N \, \mathrm{d}S \leqslant \int_V \sigma_{ij}^{e^N} \varepsilon_{ij}^{e^N} \, \mathrm{d}V \tag{3.22}$$

where $\sigma_{ij}^{e^N}$ and $\varepsilon_{ij}^{e^N}$, represent the stress and strain associated with the displacement e_i^N, respectively.

However, the right-hand side in Eq. (3.22) can be rewritten

$$\int_V \sigma_{ij}^w \varepsilon_{ij}^w \, \mathrm{d}V = \int_V \left(\sum_{m=1}^{N} c_m \sigma_{ij}^{(m)} + \sigma_{ij}^{e^N} \right) \left(\sum_{n=1}^{N} c_n \varepsilon_{ij}^{(n)} + \varepsilon_{ij}^{e^N} \right) \mathrm{d}V$$

or

$$\int_V \sigma_{ij}^w \varepsilon_{ij}^w \, \mathrm{d}V = \sum_{n=1}^{N} \lambda_n c_n^2 + \int_V \sigma_{ij}^{e^N} \varepsilon_{ij}^{e^N} \, \mathrm{d}V$$

and therefore the term $\int_V \sigma_{ij}^{e^N} \varepsilon_{ij}^{e^N} \, \mathrm{d}V$ is bounded from above.

By virtue of the inequality (3.22) and the infinite growth of λ_{N+1}

$$E^N = \int_S \varphi_{ij} e_i^N e_j^N \, \mathrm{d}S \to 0 \quad \text{for } N \to \infty$$

This proves

$$\int_S \varphi_{ij} w_i w_j \, \mathrm{d}S - \sum_{n=1}^{N} c_n^2 = 0 \quad \text{for } N \to \infty$$

and therefore establishes the completeness of the eigenfunctions. Consequently, we can write the *Parseval equality*

$$\int_S \varphi_{ij} w_i w_j \, \mathrm{d}S = \sum_{n=1}^{\infty} c_n^2 \quad \square \tag{3.23}$$

The completeness of eigendeformations means that we can approximate every piecewise continuous integrable function defined on the boundary S in the mean to any desired degree of accuracy by choosing a sufficient number of eigenmodes.

From the discussion in this section, one can conclude that the metric space of fundamental boundary eigenmodes is a Hilbert space.

Formally then the Fourier series or fundamental expansion for every L_2-function $\boldsymbol{w}$ may be written

$$\boldsymbol{w} = \sum_{n=1}^{\infty} c_n \boldsymbol{u}^{(n)} \quad \text{in } V \cup S \tag{3.24}$$

where c_n is the Fourier coefficient defined by

$$c_n = \int_S \boldsymbol{w} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{u}^{(n)} \, \mathrm{d}S = \int_S \varphi_{ij} w_i u_j^{(n)} \, \mathrm{d}S \tag{3.25}$$

We consider the continuation of the function $\boldsymbol{w}$ in the domain such that it is the elastic solution to an elasticity problem with the boundary displacement $\boldsymbol{w}$ specified. In general, $\boldsymbol{w}$ here can be discontinuous on the boundary.

In the theory of fundamental eigenmodes, we considered φ_{ij} as positive definite. We should emphasize this property is a necessary condition for having a complete set of eigenmodes for representing all given elastostatic problems. If φ_{ij} is positive semi-definite on some parts of the boundary, then the eigenmodes follow all the previously mentioned theorems, but are complete for representing only those problems in which $\boldsymbol{t}$ is normal to any principal directions corresponding to zero principal value of φ_{ij} on those boundary segments. One interesting case is when $\varphi_{ij} = 0$ on some parts of the boundary. The eigenmodes are then complete for representing only problems with $\boldsymbol{t} = 0$ on those parts of the boundary.

3.9. Convergence of generalized Fourier series

We know that the fundamental expansion converges in the mean to $\boldsymbol{w}$ on the boundary. Does the expansion converge to $\boldsymbol{w}$ on the boundary and in the domain?

The following theorem gives convergence behavior for all points.

Theorem 7. *The generalized Fourier series* (3.24) *converges at each point x in the domain to $\boldsymbol{w}(x)$. If $\boldsymbol{w}$ is piecewise regular on the boundary S, then the generalized Fourier series* (3.24) *converges at each point x on the boundary S to the principal mean value $\hat{\boldsymbol{w}}$* (*Appendix B*).

The degree of continuity of $\boldsymbol{w}$ determines the speed of decrease of c_n for higher modes. The coefficients c_n decrease faster when the function $\boldsymbol{w}$ is continuous. When $\boldsymbol{w}$ has a discontinuity at one or more points, the speed at which the coefficients decrease is not as fast.

3.10. Generalized Fourier series or fundamental expansion for $\boldsymbol{u}$, $\boldsymbol{t}$ *and* $\boldsymbol{t}^{\varphi}$

The generalized Fourier series or fundamental expansion for displacement $\boldsymbol{u}$ is

$$\boldsymbol{u} = \sum_{n=1}^{\infty} A_n \boldsymbol{u}^{(n)} \quad \text{in } V \cup S \tag{3.26}$$

where for normalized eigenmodes A_n is the Fourier coefficient defined by

$$A_n = \int_S \boldsymbol{u} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{u}^{(n)} \, \mathrm{d}S = \int_S \varphi_{ij} u_i u_j^{(n)} \, \mathrm{d}S \tag{3.27}$$

The Parseval equality becomes

$$\sum_{n=1}^{\infty} A_n^2 = \int_S \varphi_{ij} u_i u_j \, \mathrm{d}S = \|\boldsymbol{u}\|^2 \tag{3.28}$$

Theorem 8. *The fundamental expansion converges uniformly to* $\boldsymbol{u}$ *in the domain V and boundary S. The expansion for first and higher order derivatives of* $\boldsymbol{u}$ *converge uniformly and absolutely in the domain V.*

What can be said concerning traction on the boundary? For the gradient of deformation, we can write

$$\frac{\partial u_i}{\partial x_j} = \sum_{n=1}^{\infty} A_n \frac{\partial u_i^{(n)}}{\partial x_j} \quad \text{in } V \cup S$$

For strain and stress we have the following

$$\varepsilon_{ij} = \sum_{n=1}^{\infty} A_n \varepsilon_{ij}^{(n)} \quad \text{in } V \cup S \tag{3.29}$$

$$\sigma_{ij} = \sum_{n=1}^{\infty} A_n \sigma_{ij}^{(n)} \quad \text{in } V \cup S \tag{3.30}$$

and for traction on the boundary S

$$t_i = \sum_{n=1}^{\infty} A_n t_i^{(n)} \quad \text{on } S \tag{3.31}$$

From the fundamental boundary condition $t_i^{(n)} = \lambda_n \varphi_{ij} u_j^{(n)}$. Then on S

$$t_i = \varphi_{ij} \sum_{n=1}^{\infty} A_n \lambda_n u_j^{(n)} \tag{3.32}$$

By using the weighted traction $\boldsymbol{t}^{\varphi}$ defined in Eq. (3.16)

$$t_i = \varphi_{ij} t_j^{\varphi} \tag{3.16}$$

we can write

$$t_i^{\varphi} = \sum_{n=1}^{\infty} A_n \lambda_n u_i^{(n)} \quad \text{on } S$$

or

$$\boldsymbol{t}^{\varphi} = \sum_{n=1}^{\infty} A_n \lambda_n \boldsymbol{u}^{(n)} \quad \text{on } S \tag{3.33}$$

In this equation for boundary points the series converges to the principal mean value of the weighted traction. What is the meaning of this series for internal points? The value of this series is the elastic solution

$$\boldsymbol{v} = \sum_{n=1}^{\infty} A_n \lambda_n \boldsymbol{u}^{(n)} \quad \text{in } V \cup S$$

whose boundary value is $\boldsymbol{t}^{\varphi}$.

As we discussed before, if $\boldsymbol{t}^{\varphi}$ is piecewise continuous on the boundary S, the formal expansion (3.33) converges uniformly to $\boldsymbol{t}^{\varphi}$ in every closed set on S containing no discontinuity. This means the Nth partial sum of the formal expansion of $\boldsymbol{t}^{\varphi}$

$$\boldsymbol{t}_N^{\varphi}(x) = \sum_{n=1}^{N} A_n \lambda_n \boldsymbol{u}^{(n)} \quad \text{on } S \tag{3.34}$$

cannot approach the function $\boldsymbol{t}^{\varphi}$ uniformly over any set containing a point or line of discontinuity of $\boldsymbol{t}^{\varphi}$. It is seen that the series oscillates near to the discontinuity. This is a generalized form of *Gibbs' phenomenon* which has been studied in detail in trigonometric Fourier series and integrals (e.g., Carslaw (1950)).

For physical problems we assume that $\boldsymbol{u}$ is continuous everywhere, but that $\boldsymbol{t}$ can be piecewise continuous. In practice we try to choose φ_{ij} such that $\boldsymbol{t}^{\varphi}$ becomes piecewise regular. The expansions for $\boldsymbol{u}$ and $\boldsymbol{t}^{\varphi}$ are complete and converge in the mean. In general every vectorial function $\boldsymbol{w}(x)$ defined on the boundary S can be expanded in terms of boundary eigenmodes if and only if $\boldsymbol{w}(x)$ is mean square integrable (L_2-function) on the boundary with respect to φ_{ij}.

By using Eqs. (3.11), (3.12) and (3.27) we derive the following expression for A_n

$$A_n = \frac{1}{\lambda_n} \int_V \sigma_{ij} \varepsilon_{ij}^{(n)} \, \mathrm{d}V \quad \text{for } \lambda_n \neq 0 \tag{3.35}$$

or

$$\int_V \sigma_{ij} \varepsilon_{ij}^{(n)} \, \mathrm{d}V = \lambda_n A_n$$

As we said, the displacement $\boldsymbol{u}$ is continuous everywhere. Now we add the requirement for strain energy to be bounded. We derive an expression for strain energy in terms of eigenenergies. By using expansions the strain energy can be written as

$$\mathscr{U} = \frac{1}{2} \int_S \left[\sum_{n=1}^{\infty} \lambda_n A_n \varphi_{ij} u_j^{(n)} \right] \left[\sum_{m=1}^{\infty} A_m u_i^{(m)} \right] \mathrm{d}S$$

Using the orthogonality property

$$\mathscr{U} = \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n A_n^2 \int_S \varphi_{ij} u_i^{(n)} u_j^{(n)} \, \mathrm{d}S \tag{3.36a}$$

If the eigenmodes are normalized, then

$$\mathscr{U} = \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n A_n^2 \tag{3.36b}$$

which means that the strain energy can be expressed as an infinite series of the eigenenergies.

Let us also obtain a new expression for the Rayleigh quotient. From Eq. (3.2), we have

$$R\{\boldsymbol{u}\} = \frac{\int_S t_i u_i \, \mathrm{d}S}{\int_S \varphi_{ij} u_i u_j \, \mathrm{d}S} = \frac{2\mathscr{U}}{\|\boldsymbol{u}\|^2}$$

and then by substituting Eqs. (3.36b) and (3.28) in the right-hand side, we derive

$$R\{\boldsymbol{u}\} = \frac{\sum_{n=1}^{\infty} \lambda_n A_n^2}{\sum_{n=1}^{\infty} A_n^2} \tag{3.37}$$

3.11. Behavior of boundary eigensolutions and Fourier coefficients

We should mention that as the eigenvalues become larger, there are a greater number of oscillations in the associated eigenmodes on the boundary. This means that the number of zeros of the eigenmodes on the boundary increases for higher modes. This is the result of the orthogonality of higher eigenmodes to lower eigenmodes.

It can also be shown that similar to other eigenvalue problems (Courant and Hilbert, 1953) by assuming that the eigenmodes are normalized such that

$$\int_S \varphi_{ij} u_i^{(n)} u_j^{(n)} \, \mathrm{d}S = 1$$

there is a number $C > 0$ independent of n such that

$$|\boldsymbol{u}^{(n)}| = \sqrt{u_i^{(n)} u_i^{(n)}} < C \tag{3.38}$$

where $|\boldsymbol{u}^{(n)}|$ is the magnitude of the vector $\boldsymbol{u}^{(n)}$.

It is seen that the order of eigenvalues for higher modes are independent of the tensor weight function Φ and

$$\lambda_n = \begin{cases} \mathrm{O}(n) & \text{in 2-D as } n \to \infty \\ \mathrm{O}(\sqrt{n}) & \text{in 3-D as } n \to \infty \end{cases} \tag{3.39}$$

In the theory of elasticity the displacement field $\boldsymbol{u}$ is continuous everywhere, even on the boundary. If φ_{ij} is such that $\boldsymbol{u}$ is an L_2-function with respect to φ_{ij}, then the Fourier series for acceptable $\boldsymbol{u}$ is

$$\boldsymbol{u} = \sum_{n=1}^{\infty} A_n \boldsymbol{u}^{(n)} \quad \text{in } V \cup S$$

where A_n is the Fourier coefficient defined by

$$A_n = \int_S \boldsymbol{u} \cdot \Phi \cdot \boldsymbol{u}^{(n)} \, \mathrm{d}S = \int_S \varphi_{ij} u_i u_j^{(n)} \, \mathrm{d}S$$

with

$$\sum_{n=1}^{\infty} A_n^2 = \int_S \varphi_{ij} u_i u_j \, \mathrm{d}S < \infty$$

If φ_{ij} is such that $\boldsymbol{t}^{\varphi}$ is an L_2-function with respect to φ_{ij}, then

$$\boldsymbol{t}^{\varphi} = \sum_{n=1}^{\infty} A'_n \boldsymbol{u}^{(n)} \quad \text{in } V \cup S$$

where

$$A'_n = \lambda_n A_n$$

It is obvious that

$$\left(|A'_n| - \frac{1}{\lambda_n}\right)^2 \geqslant 0 \quad \text{for } n > n_{\mathrm{R}}$$

or

$$A'^2_n - 2\frac{|A'_n|}{\lambda_n} + \frac{1}{\lambda_n^2} \geqslant 0 \quad \text{for } n > n_{\mathrm{R}}$$

where n_{R} is the number of rigid body modes. Then

$$2\frac{|A'_n|}{\lambda_n} \leqslant A'^2_n + \frac{1}{\lambda_n^2} \qquad \text{for } n > n_{\mathrm{R}}$$

or for coefficient A_n

$$2|A_n| \leqslant A'^2_n + \frac{1}{\lambda_n^2} \qquad \text{for } n > n_{\mathrm{R}}$$

Because the boundary weighted traction vector $\boldsymbol{t}^{\varphi}$ is an L_2-function with respect to φ_{ij}

$$\int_S \varphi_{ij} t_i^{\varphi} t_j^{\varphi}\, \mathrm{d}S = \int_S t_i t_i^{\varphi}\, \mathrm{d}S = \sum_{n=1}^{\infty} A'^2_n < \infty$$

Then, by knowing $\sum_{n=n_{\mathrm{R}}+1}^{\infty}(1/\lambda_n^2) < \infty$ from the flexibility concept which will be discussed later, we see that the series

$$\sum_{n=n_{\mathrm{R}}+1}^{\infty} A'^2_n + \frac{1}{\lambda_n^2}$$

is convergent. Therefore the series $\sum_{n=1}^{\infty} |A_n|$ is convergent, which means $\sum_{n=1}^{\infty} A_n$ is absolutely convergent.

By knowing that $|\boldsymbol{u}^{(n)}| = (u_i^{(n)} u_i^{(n)})^{1/2} < C$ no matter the value of n, we can see that the higher modes have less contribution to Fourier expansions. This is a very important result in computational mechanics.

Strain energy is bounded, then

$$\int_S t_i u_i\, \mathrm{d}S = \int_S \varphi_{ij} u_i t_j^{\varphi}\, \mathrm{d}S = \sum_{n=1}^{\infty} \lambda_n A_n^2 < \infty$$

which means that as n increases, $\lambda_n A_n^2 \to 0$. In other words A_n decreases faster than $1/(\lambda_n)^{1/2}$. In practice we choose φ_{ij} such that

$$\int_S t_i t_i^{\varphi}\, \mathrm{d}S = \int_S \varphi_{ij} t_i^{\varphi} t_j^{\varphi}\, \mathrm{d}S = \sum_{n=1}^{\infty} \lambda_n^2 A_n^2 < \infty$$

which means A_n decreases faster than $1/\lambda_n$ such that

$$\lambda_n A_n \to 0 \quad \text{as } n \to \infty$$

We will prove that the series $\sum_{n=1}^{\infty} A_n \boldsymbol{u}^{(n)}$ is absolutely convergent. We see that

$$|A_n \boldsymbol{u}^{(n)}| \leqslant C|A_n|$$

and the series $\sum_{n=1}^{\infty} C|A_n|$ is convergent. Then from the *Weierstrass' M-test* theorem the series $\sum_{n=1}^{\infty} A_n \boldsymbol{u}^{(n)}$ converges uniformly and absolutely on $V \cup S$.

What is the impact of continuity in these expansions? The behavior of the coefficients and the uniformity of convergence is the answer.

We can see the expansion (3.26) for $\boldsymbol{u}$ converges faster than expansion (3.31) for $\boldsymbol{t}$. By choosing a suitable Φ to make $\boldsymbol{t}^{\varphi}$ piecewise regular (i.e., bounded), we increase the rate of convergence of its expansion (3.33) beyond that achieved by the expansion (3.31) for $\boldsymbol{t}$. This is an important conclusion which we use in computational mechanics in Part II.

In Part II, we show that the computational methods such as finite element and boundary element methods follow the fundamental boundary eigenmodes theory with $\varphi_{ij} = \delta_{ij}$. If the number of boundary degrees of freedom in the model is N, solutions are based on N eigenmodes. If we assume that we have exact eigenmodes, the approximate expansion for $\boldsymbol{u}$ and $\boldsymbol{t}$ are

$$\boldsymbol{u}_N = \sum_{n=1}^{N} A_n \boldsymbol{u}^{(n)}, \quad V \cup S \tag{3.40}$$

and

$$\boldsymbol{t}_N = \sum_{n=1}^{N} A_n \lambda_n \boldsymbol{u}^{(n)} \quad \text{on } S \tag{3.41}$$

These expansions approximate the quantities in mean square value. However, there is a difference in the character of the expansions. The expansion (3.40) converges uniformly to $\boldsymbol{u}$, because $\boldsymbol{u}$ is continuous everywhere. In contrast the expansion (3.41) does not converge to $\boldsymbol{t}$ uniformly at points on the boundary where $\boldsymbol{t}$ is discontinuous. Tractions may have a singularity at these points. The expansion (3.41) is a summation of a finite number N of continuous eigenmodes. From analysis we know this summation is a continuous function on the boundary. If N approaches infinity, the summation can be discontinuous and capture the singularity of tractions. The finite series (3.41) is a continuous function which has bounded values and cannot show the singularity. The remedy is to choose such a function Φ that makes $\boldsymbol{t}^{\varphi}$ piecewise regular, which means still that it will have the discontinuity but now exhibit only bounded values. The approximate expansion for $\boldsymbol{t}^{\varphi}$ is

$$\boldsymbol{t}_N^{\varphi}(x) = \sum_{n=1}^{N} A_n \lambda_n \boldsymbol{u}^{(n)} \quad \text{on } S \tag{3.42}$$

Keep in mind that the fundamental boundary eigenmodes in Eqs. (3.41) and (3.42) are different. The former are obtained with respect to $\varphi_{ij} = \delta_{ij}$ on the boundary, while the latter are based on a special Φ which takes care of the singularity. The expansion (3.42) has Gibbs' oscillations near the discontinuity, but it oscillates about finite values. These ideas will be used in Part II to develop new computational formulations.

3.12. Analyticity of eigenmodes for $\varphi_{ij} = \delta_{ij}$ and non-analytic eigenmodes

Let us consider the case of the fundamental boundary eigenproblem for $\varphi_{ij} = \delta_{ij}$. The interesting character of this case is that the eigenmodes are analytic on the boundary. The reason is that the fundamental boundary condition becomes

$$\boldsymbol{t} = \lambda \boldsymbol{u} \quad \text{on } S$$

where $\boldsymbol{u}$ is an eigenmode. The traction $\boldsymbol{t}$ of the eigenmodes is bounded, because $\boldsymbol{u}$ is bounded. Now we can consider $\boldsymbol{t}$ as a vector displacement field continued in the domain such that it is an elastic solution which satisfies the boundary condition

$$\boldsymbol{t}^{[1]} = \lambda \boldsymbol{t} = \lambda^2 \boldsymbol{u} \quad \text{on } S$$

where $\boldsymbol{t}^{[1]}$ is the traction of displacement field $\boldsymbol{t}$ on the boundary. Again we continue $\boldsymbol{t}^{[1]}$ in the domain such that it is the displacement of an elastic solution. We see

$$\boldsymbol{t}^{[2]} = \lambda \boldsymbol{t}^{[1]} = \lambda^3 \boldsymbol{u} \quad \text{on } S$$

where $\boldsymbol{t}^{[2]}$ is the traction of displacement field $\boldsymbol{t}^{[1]}$ on the boundary. By repeating this process

$$\boldsymbol{t}^{[p]} = \lambda \boldsymbol{t}^{[p-1]} = \lambda^{p+1} \boldsymbol{u} \quad \text{on } S$$

Thus we see that $\boldsymbol{t}^{[p]}$ is related to the $(p+1)$th derivative of $\boldsymbol{u}$ which is bounded even on the boundary. This proves that a Taylor expansion is possible in the neighborhood of boundary points no matter whether the boundary is smooth or non-smooth and the eigenmodes are analytic on the boundary S. Although this is an interesting property, it is not suitable for the practical solution of non-smooth problems involving, for example, cracked and notched bodies.

In Part II, we will show that the traditional boundary element and traction oriented finite element methods follow the theory of fundamental eigensolutions with $\varphi_{ij} = \delta_{ij}$. This means that the solutions approximate the singular stress distribution in terms of a partial sum of a finite number of approximated eigenmodes. But this partial sum is a continuous expression which cannot capture the behavior at singular points. The partial sum converges far from singular points which can be explained by the Saint-Venant theorem, a property common to all elliptic problems.

Although the analyticity of the eigenmodes with $\varphi_{ij} = \delta_{ij}$ is an important property as mentioned previously, alternative weight functions are preferable in practice when dealing with non-smooth problems. Instead for these problems, we may utilize non-analytic eigenmodes obtained by choosing suitable discontinuous and even singular φ_{ij}. For example, appropriate weight functions can be constructed from a local analysis around singular points. In 2-D, Williams (1952) has given the asymptotic behavior of stresses around a notch tip as $r^{\gamma-1}$ where r is the radial distance from singular point and γ is a parameter depending on geometry, material properties, and type of boundary conditions. If we take $\varphi_{ij} = r^{\gamma-1}\delta_{ij}$, then the eigenmodes corresponding to this weight function are non-analytic and help to promote convergence in a systematic way. These ideas will be discussed in more detail in Part II.

3.13. Fundamental coefficients for BVPs

Dirichlet problem. Assume the value of displacement $\boldsymbol{u}$ is prescribed everywhere on the boundary such that $\boldsymbol{u} = \boldsymbol{f}(x)$ on S, where $\boldsymbol{f}(x)$ is an L_2-function with respect to Φ. Using Eq. (3.27), we obtain the fundamental coefficients as

$$A_n = \int_S \boldsymbol{f} \cdot \Phi \cdot \boldsymbol{u}^{(n)}\, \mathrm{d}S = \int_S \varphi_{ij} f_i u_j^{(n)}\, \mathrm{d}S \tag{3.43}$$

assuming orthonormalized eigensolutions.

Neumann problem. Assume the value of traction $\boldsymbol{t}$ is prescribed everywhere on the boundary such that $\boldsymbol{t} = \boldsymbol{g}(x)$ on S, where $\boldsymbol{g}(x)$ is a mean square integrable function satisfying the equilibrium conditions

$$\int_S \boldsymbol{g}(x)\, \mathrm{d}S = \boldsymbol{0}$$

and

$$\int_S \boldsymbol{x} \times \boldsymbol{g}(x)\, \mathrm{d}S = \boldsymbol{0}$$

From the expansion (3.32), we obtain

$$A_n = \frac{1}{\lambda_n} \int_S \boldsymbol{g} \cdot \boldsymbol{u}^{(n)}\, \mathrm{d}S = \frac{1}{\lambda_n} \int_S g_i u_i^{(n)}\, \mathrm{d}S \quad \text{for } \lambda_n \neq 0 \tag{3.44}$$

assuming orthonormalized eigensolutions. Coefficients corresponding to $\lambda_n = 0$ are not determined, because we can add an arbitrary rigid body motion to the solution in the Neumann problem.

Mixed problem. In this case, the value of $\boldsymbol{t}$ is specified on some portion of the boundary and the value of $\boldsymbol{u}$ is specified on the rest of the boundary. In Section 1, we classified this problem as non-smooth, along with general problems involving corners, notches and cracks. The common feature of all of these problems is singularity of the solution. Many practical engineering problems are of this type. We may still use the relationships inherent in Eqs. (3.43) and (3.44), but we cannot obtain a closed form solution for the fundamental coefficients in general. This case is related to methods such as dual series equations for bounded domains and dual integral equations for unbounded domains (Sneddon, 1966), Hilbert problem (Muskhelishvili, 1953) and the Wiener–Hopf technique (Noble, 1958). The numerical solution of mixed problems is addressed in Hadjesfandiari and Dargush (2001b) and Part II by introducing new boundary element and finite element formulations.

We should emphasize that, in general, the fundamental boundary eigensolutions are not available in closed form since these are dependent on an arbitrary tensor function and the shape of the domain. In reality finding the coefficients in closed form is almost impossible. However, in the case of circular and spherical domains for certain weight functions we can derive these modes. Section 4 provides the details for a circular domain.

More importantly, the mathematical concepts direct us toward a better understanding of elastic boundary value problems and provide a useful strategy for computational mechanics.

3.14. Virtual work theorem

Now we try to give the weak formulation for fundamental boundary eigensolutions. This formulation can be derived from the principle of virtual work, which can be written

$$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,\mathrm{d}V = \int_S t_i\,\delta u_i\,\mathrm{d}S \tag{3.45}$$

with the variation δu_i considered on the entire boundary S. Notice that the boundary on the right-hand side of Eq. (3.45) is the entire boundary S. The variation of δu_i is now incompatible with the essential boundary conditions. By considering $t_i = \varphi_{ij} t_j^{\varphi}$, we have

$$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,\mathrm{d}V = \int_S \varphi_{ij} t_j^{\varphi}\,\delta u_i\,\mathrm{d}S \tag{3.46}$$

By inserting the fundamental boundary condition $t_i^{\varphi} = \lambda u_j$, we have

$$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,\mathrm{d}V = \lambda \int_S \varphi_{ij} u_j\,\delta u_i\,\mathrm{d}S \tag{3.47}$$

This is the weak formulation for fundamental boundary eigensolutions. This result can, of course, be used to formulate finite element methods. In Part II, a discretized version of Eq. (3.47) is used to develop a finite element formulation for the fundamental eigenproblem. Furthermore, the above variational framework (3.46) leads to the development of a traction-oriented finite element method that has distinct advantages over existing approaches for the solution of general smooth and non-smooth boundary value problems. Details of this finite element formulation and the associated numerical implementation are also presented in Hadjesfandiari (1998) and Part II. In the next section by defining the boundary stiffness kernel we give the virtual work formulation exclusively in terms of boundary integrals.

3.15. Flexibility and stiffness kernels

Considering two points x and ξ belonging to the domain V, we can define the tensorial kernel

$$p_{ij}(\xi,x) = \sum_{n=n_{\mathrm{R}}+1}^{\infty} \frac{u_i^{(n)}(\xi)u_j^{(n)}(x)}{\lambda_n}, \quad x,\xi \in V \cup S \tag{3.48}$$

Here, for simplicity we consider eigenmodes corresponding to the case $\varphi_{ij} = \varphi\delta_{ij}$. In this definition, we exclude the rigid body eigenmodes corresponding to $\lambda_n = 0$ with multiplicity n_{R}. Then from Eq. (3.48) along with Eq. (3.32), we see that for every acceptable deformation $\boldsymbol{u}$ which satisfies equilibrium

$$\sum_{n=n_{\mathrm{R}}+1}^{\infty} A_n u_i^{(n)}(\xi) = \int_S p_{ij}(\xi,x)t_j(x)\,\mathrm{d}S(x), \quad \xi \in V \cup S$$

or

$$u_i(\xi) - u_i^{\mathrm{R}}(\xi) = \int_S p_{ij}(\xi,x)t_j(x)\,\mathrm{d}S(x), \quad \xi \in V \cup S \tag{3.49}$$

where u_i^{R} is the total rigid body motion. Actually by noticing that the above integral is the solution to a Neumann problem, the presence of rigid body motion is justified.

In analogy with structural mechanics, the kernel $p_{ij}(\xi,x)$ has the character of flexibility. Therefore we call it the *boundary flexibility kernel* when the points x and ξ are on the boundary.

We can see for strain energy

$$\mathscr{U} = \frac{1}{2}\int_S t_i(\xi)u_i(\xi)\,\mathrm{d}S(\xi) = \frac{1}{2}\int_S\int_S p_{ij}(\xi,x)t_j(x)t_i(\xi)\,\mathrm{d}S(x)\,\mathrm{d}S(\xi) \tag{3.50}$$

Due to self equilibrium of $\boldsymbol{t}$, the rigid body motion in Eq. (3.50) vanishes.

The complementary virtual work is

$$\int_S\int_S p_{ij}(x,y)t_j(y)\,\delta t_i(x)\,\mathrm{d}S(y)\,\mathrm{d}S(x) = \int_S u_i(x)\,\delta t_i(x)\,\mathrm{d}S(x) - \int_S u_i^{\mathrm{R}}(x)\,\delta t_i(x)\,\mathrm{d}S(x)$$

or

$$\int_S\int_S p_{ij}(x,y)t_j(y)\,\delta t_i(x)\,\mathrm{d}S(y)\,\mathrm{d}S(x) = \int_S u_i(x)\,\delta t_i(x)\,\mathrm{d}S(x) - \sum_{m=1}^{n_{\mathrm{R}}} A_m \int_S u_i^{(m)}(x)\,\delta t_i(x)\,\mathrm{d}S(x) \tag{3.51}$$

in which $\delta\boldsymbol{t}$ is an admissible variation of traction $\boldsymbol{t}$. Alternatively, in terms of the weighted traction, we have

$$\int_S\int_S p_{ij}(x,y)\varphi(y)t_j^{\varphi}(y)\varphi(x)\,\delta t_i^{\varphi}(x)\,\mathrm{d}S(y)\,\mathrm{d}S(x) = \int_S u_i(x)\varphi(x)\,\delta t_i^{\varphi}(x)\,\mathrm{d}S(x) - \sum_{m=1}^{n_{\mathrm{R}}} A_m \times \int_S u_i^{(m)}(x)\,\delta t_i^{\varphi}(x)\,\mathrm{d}S(x) \tag{3.52}$$

We can see that for, $\xi \in S$, $p_{ij}(\xi,x)$ is an elastic solution

$$C_{ijkl}p_{rk,lj}(\xi,x) = 0 \quad \text{in } V \text{ and } \xi \in S \tag{3.53a}$$

such that on the boundary the traction

$$t_{ri}^{p}(\xi,x) = C_{ijkl}p_{rk,l}(\xi,x)n_j(x) = \delta_S(x,\xi)\delta_{ri} - \varphi(x)\sum_{m=1}^{n_{\mathrm{R}}} u_r^{(m)}(\xi)u_i^{(m)}(x) \quad \text{on } S \tag{3.53b}$$

where $\delta_S(x,\xi)$ is a Dirac-delta function on the boundary S defined in Appendix C. The first term related to $\delta_S(x,\xi)$ represents the unit boundary concentrated forces and the rigid body terms represent boundary distributed tractions for maintaining equilibrium as

$$\int_S t^p_{ri}(\xi,x)\,\mathrm{d}S(x) = 0$$

and

$$\int_S \varepsilon_{ijk} x_j t^p_{rk}(\xi,x)\,\mathrm{d}S(x) = 0$$

Detailed expressions for the summation over the rigid body modes are presented in Appendix A. It should be mentioned that in Eqs. (3.53a) and (3.53b) derivatives are with respect to x.

The solution is given by Eq. (3.48) for ξ on the boundary which by continuation defines this kernel for all points x and ξ in the domain $V \cup S$. The kernel $p_{ij}(\xi,x)$ has a weak singularity like that of the $G_{ij}(\xi,x)$ kernel in the boundary integral method when ξ is on the boundary. Thus, the kernel $p_{ij}(\xi,x)$ is an L_2-function which means

$$\int_S \int_S p_{ij}(\xi,x) p_{ij}(\xi,x)\,\mathrm{d}S(x)\,\mathrm{d}S(\xi) < \infty$$

By using the degenerate expansion of $p_{ij}(\xi,x)$, we can easily see

$$\sum_{n=n_\mathrm{R}+1}^{\infty} \frac{1}{\lambda_n^2} < \infty \tag{3.54}$$

which means the series $\sum_{n=n_\mathrm{R}+1}^{\infty} 1/\lambda_n^2$ is convergent. (Note that this result was used earlier to prove that $\sum_{n=1}^{\infty} A_n$ is absolutely convergent.)

Now consider ξ on the boundary S and look for an elastic solution $m_{ij}(\xi,x)$ that satisfies

$$C_{ijkl} m_{rk,lj}(\xi,x) = 0 \quad \text{in } V \text{ and } \xi \in S \tag{3.55a}$$

such that on the boundary the displacement

$$\varphi(x) m_{ij}(\xi,x) = \delta_S(x,\xi)\delta_{ij} \quad \text{on } S \tag{3.55b}$$

By using the reciprocal theorem between $m_{ij}(\xi,x)$ and an elastic solution u_i

$$\int_S [m_{ri}(\xi,x) t_i(x) - t^m_{ri}(\xi,x) u_i(x)]\,\mathrm{d}S(x) = 0$$

where

$$t^m_{ri}(\xi,x) = C_{ijkl} m_{rk,l(x)}(\xi,x) n_j(x), \quad x \in S$$

and by using Eq. (3.55b)

$$\int_S \left[\frac{1}{\varphi(x)} \delta_S(x,\xi)\delta_{ri} t_r(x) - C_{ijkl} m_{rk,l}(\xi,x) n_j(x) u_r(x)\right] \mathrm{d}S(x) = 0$$

Then

$$\hat{t}^{\varphi}_i(\xi) = \int_S C_{ijkl} m_{kr,l}(\xi,x) n_j(x) u_r(x)\,\mathrm{d}S(x), \quad \xi \in S \tag{3.56}$$

In analogy with structural mechanics, the kernel $C_{ijkl} m_{rk,l}(\xi,x) n_j(x)$ exhibits the character of stiffness. Therefore, we call $k_{ij}(\xi,x)$ the *boundary stiffness kernel*, where

$$\varphi(x)k_{ri}(\xi,x) = C_{ijkl}m_{rk,l}(\xi,x)n_j(x), \quad x,\xi \in S \tag{3.57}$$

Then

$$\hat{t}_i^{\varphi}(\xi) = \int_S \varphi(x)k_{ij}(\xi,x)u_j(x)\,\mathrm{d}S(x), \quad \xi \in S \tag{3.58}$$

Now let us derive expressions for the kernels $m_{ij}(\xi,x)$ and $k_{ij}(\xi,x)$ in terms of eigenmodes. By knowing $m_{ij}(\xi,x)$ is an elastic solution although not an L_2-function, we formally write the expression

$$m_{ij}(\xi,x) = \sum_{n=1}^{\infty} a_i^{(n)}(\xi)u_j^{(n)}(x), \quad \xi \in S$$

with

$$a_i^{(n)}(\xi) = \int_S \varphi(x)m_{ij}(\xi,x)u_j^{(n)}(x)\,\mathrm{d}S(x), \quad \xi \in S$$

or

$$a_i^{(n)}(\xi) = \int_S \delta_S(x,\xi)\delta_{ij}u_j^{(n)}(x)\,\mathrm{d}S(x), \quad \xi \in S$$

and finally

$$a_i^{(n)}(\xi) = u_i^{(n)}(\xi)$$

Therefore the degenerate form can be written

$$m_{ij}(\xi,x) = \sum_{n=1}^{\infty} u_i^{(n)}(\xi)u_j^{(n)}(x), \quad \xi \in S \tag{3.59}$$

Now, we release the constraint $\xi \in S$ and let ξ be anywhere in the domain, so that we have a continuation for $m_{ij}(\xi,x)$ as

$$m_{ij}(\xi,x) = \sum_{n=1}^{\infty} u_i^{(n)}(\xi)u_j^{(n)}(x), \quad x,\xi \in V \cup S \tag{3.60}$$

It is seen that for every point $\xi \in V \cup S$ we have

$$u_i(\xi) = \int_S \varphi(x)m_{ij}(\xi,x)u_j(x)\,\mathrm{d}S(x), \quad \xi \in V \cup S \tag{3.61}$$

This is the solution to the Dirichlet problem in elastostatics. The kernel $m_{ij}(\xi,x)$ is similar to the Poisson kernel in the potential problem which exists for the circle and sphere in closed form (e.g., Garabedian (1988)). The kernel is bounded when x and ξ are internal points and is zero when both x and ξ are on the boundary but not coincident. It has a strong singularity when one of the points is on the boundary and the other point approaches to it from inside the domain. The singularity of $m_{ij}(\xi,x)$ is similar to the singularity of the kernel $F_{ij}(\xi,x)$ in the boundary integral method.

From the definition of $k_{ri}(\xi,x)$ in Eq. (3.57) we obtain

$$\varphi(x)k_{ri}(\xi,x) = C_{ijkl}m_{rk,l}(\xi,x)n_j(x), \quad x,\xi \in S \tag{3.57}$$

$$\varphi(x)k_{ri}(\xi,x) = C_{ijkl}\sum_{n=1}^{\infty} u_r^{(n)}(\xi)u_{k,l}^{(n)}(x)n_j(x), \quad x,\xi \in S$$

but $C_{ijkl}u_{k,l}^{(n)}(x)n_j(x) = t_i^{(n)}(x) = \lambda_n\varphi(x)u_i^{(n)}(x)$. Therefore

$$k_{ri}(\xi,x) = \sum_{n=1}^{\infty} \lambda_n u_r^{(n)}(\xi) u_i^{(n)}(x), \quad x,\xi \in S \tag{3.62}$$

It is easily seen that for every L_2-function $\boldsymbol{u}$,

$$\int_S \varphi(x) k_{ij}(\xi,x) u_j(x)\,\mathrm{d}S(x) = \sum_{n=1}^{\infty} \lambda_n A_n u_i^{(n)}(\xi), \quad \xi \in S$$

and we derive Eq. (3.58) again

$$\int_S \varphi(x) k_{ij}(\xi,x) u_j(x)\,\mathrm{d}S(x) = \hat{t}_i^{\varphi}(\xi), \quad \xi \in S \tag{3.58}$$

Now we continue the kernel in the domain by letting points x and ξ be anywhere in the domain. Thus,

$$k_{ij}(\xi,x) = \sum_{n=1}^{\infty} \lambda_n u_i^{(n)}(\xi) u_j^{(n)}(x), \quad x,\xi \in V \cup S \tag{3.63}$$

When the points x and ξ are on the boundary, the kernel $k_{ri}(\xi,x)$ is hypersingular and the integral in Eq. (3.58) is interpreted as Hadamard finite part. The singularity is similar to the singularity of the kernel $C_{ijkl}F_{rk,l(x)}(\xi,x)n_j(x)$.

For strain energy, we can now write

$$\mathscr{U} = \frac{1}{2}\int_S t_i(\xi) u_i(\xi)\,\mathrm{d}S(\xi) = \frac{1}{2}\int_S \int_S \varphi(x)\varphi(\xi) k_{ij}(\xi,x) u_i(\xi) u_j(x)\,\mathrm{d}S(x)\,\mathrm{d}S(\xi) \tag{3.64}$$

and the weak formulation or virtual work theorem in bilinear form is

$$\int_S \int_S \varphi(x)\varphi(y) k_{ij}(x,y) u_j(y)\,\delta u_i(x)\,\mathrm{d}S(y)\,\mathrm{d}S(x) = \int_S t_i(x)\,\delta u_i(x)\,\mathrm{d}S(x) \tag{3.65}$$

or in terms of the weighted traction

$$\int_S \int_S \varphi(x)\varphi(y) k_{ij}(x,y) u_j(y)\,\delta u_i(x)\,\mathrm{d}S(y)\,\mathrm{d}S(x) = \int_S \varphi(x) t_i^{\varphi}(x)\,\delta u_i(x)\,\mathrm{d}S(x) \tag{3.66}$$

This boundary integral form of virtual work is actually what the finite element method attempts to resemble as we will see in Part II.

From the expansions we observe that the kernels $p_{ij}(\xi,x)$, $m_{ij}(\xi,x)$ and $k_{ij}(\xi,x)$ are symmetric with respect to simultaneous interchange of the indices and the points x and ξ. A very interesting relation is that

$$k_{ri}(\xi,x) = C_{ijkl} m_{rk,l(x)}(\xi,x) n_j(x) = C_{rjkl} m_{ik,l(\xi)}(x,\xi) n_j(\xi), \quad x,\xi \in S \tag{3.67}$$

which is the result of $k_{ri}(\xi,x) = k_{ir}(x,\xi)$.

Now we obtain a relation between the three kernels. Let us consider the integral

$$\int_S \varphi(\xi) k_{ij}(x,\xi) p_{jk}(\xi,y)\,\mathrm{d}S(\xi) = \int_S \varphi(\xi)\left[\sum_{m=1}^{\infty} \lambda_m u_i^{(m)}(x) u_j^{(m)}(\xi)\right]\left[\sum_{n=n_R+1}^{\infty} \frac{u_j^{(n)}(\xi) u_k^{(n)}(y)}{\lambda_n}\right]\mathrm{d}S(\xi),$$
$$x,y \in V \cup S$$

or

$$\int_S \varphi(\xi) k_{ij}(x,\xi) p_{jk}(\xi,y)\,\mathrm{d}S(\xi) = \sum_{m=1}^{\infty}\sum_{n=n_R+1}^{\infty} \frac{\lambda_m}{\lambda_n} u_i^{(m)}(x) u_k^{(n)}(y) \int_S \varphi(\xi) u_j^{(m)}(\xi) u_j^{(n)}(\xi)\,\mathrm{d}S(\xi)$$

Using the orthogonality condition

$$\int_S \varphi(\xi)k_{ij}(x,\xi)p_{jk}(\xi,y)\,\mathrm{d}S(\xi) = \sum_{m=n_R+1}^{\infty} u_i^{(m)}(x)u_k^{(m)}(y) \tag{3.68}$$

which is

$$\int_S \varphi(\xi)k_{ij}(x,\xi)p_{jk}(\xi,y)\,\mathrm{d}S(\xi) = m_{ik}^D(x,y), \quad x,y \in V \cup S \tag{3.69}$$

where

$$m_{ik}^D(x,y) = m_{ik}(x,y) - m_{ik}^R(x,y) \tag{3.70}$$

with

$$m_{ik}^R(x,y) = \sum_{m=1}^{n_R} u_i^{(m)}(x)u_k^{(m)}(y)$$

For the special case when the points x and y both approach the boundary

$$\int_S \varphi(\xi)k_{ij}(x,\xi)p_{jk}(\xi,y)\,\mathrm{d}S(\xi) = \frac{1}{\varphi(x)}\delta_{ik}\delta_S(x,y) - m_{ik}^R(x,y), \quad x,y \in S \tag{3.71}$$

Alternatively, we could initiate the discussion this way. By the concept of stiffness, we propose the integral (3.58) for expressing traction in terms of boundary displacement

$$\hat{t}_i(\xi) = \int_S \varphi(x)\varphi(\xi)k_{ij}(\xi,x)u_j(x)\,\mathrm{d}S(x), \quad \xi \in S \tag{3.58}$$

In other words, the above integral is the solution to the Dirichlet problem. By viewing the direct integral equation we can expect that we are dealing with a hypersingular kernel. The fundamental boundary eigenproblem is the natural spectral analysis of this integral equation, which is written

$$\lambda\varphi(\xi)u_i(\xi) = \int_S \varphi(x)\varphi(\xi)k_{ij}(\xi,x)u_j(x)\,\mathrm{d}S(x), \quad \xi \in S$$

or

$$\lambda u_i(\xi) = \int_S \varphi(x)k_{ij}(\xi,x)u_j(x)\,\mathrm{d}S(x), \quad \xi \in S \tag{3.72}$$

This is similar to Hilbert–Schmidt theory and follows its consequences although the kernel is not an L_2-function (Kanwal, 1971). The degenerate form of the boundary stiffness kernel is

$$k_{ij}(\xi,x) = \sum_{n=1}^{\infty} \lambda_n u_i^{(n)}(\xi)u_j^{(n)}(x), \quad x,\xi \in S \tag{3.73}$$

Similarly, by the concept of flexibility, we can propose the integral (3.49) for expressing displacements in terms of boundary traction

$$u_i(\xi) - u_i^{\mathrm{R}}(\xi) = \int_S p_{ij}(\xi,x)t_j(x)\,\mathrm{d}S(x), \quad \xi \in V \cup S \tag{3.49}$$

Of course an arbitrary rigid body motion appears. In other words the above integral is the solution to the Neumann problem. By viewing the direct integral equation we can expect that we are dealing here with a weakly singular kernel. The fundamental boundary eigenproblem is again the natural spectral analysis of this integral equation, which for non-rigid body modes becomes

$$u_i(\xi) = \lambda \int_S p_{ij}(\xi, x)\varphi(x)u_j(x)\,\mathrm{d}S(x), \quad \xi \in S \tag{3.74}$$

Again, this is similar to Hilbert–Schmidt theory and follows its consequences. The degenerate form of the boundary flexibility kernel is

$$p_{ij}(\xi, x) = \sum_{n=n_\mathrm{R}+1}^{\infty} \frac{u_i^{(n)}(\xi)u_j^{(n)}(x)}{\lambda_n}, \quad x, \xi \in S \tag{3.75}$$

The tensor φ_{ij} disappears in this form although the eigenmodes depend on it.

The degenerate form of the kernels $k_{ij}(\xi, x)$ and $p_{ij}(\xi, x)$ convince us to consider λ_n and $1/\lambda_n$ as eigenstiffness and eigenflexibility, respectively.

Finally, we note that the kernels $p_{ij}(\xi, x)$, $m_{ij}(\xi, x)$ and $k_{ij}(\xi, x)$ are regular when the points x, ξ are in the domain V. They have singularity only when these points are on the boundary S.

4. Closed form boundary eigensolutions

Consider a circular isotropic elastic body with radius a in plane strain deformation. Parton and Perlin (1977) presented the general solutions for this problem. Here we present the complete set of the boundary eigensolutions in closed form. In polar coordinates the rigid body eigenmodes (type RB), corresponding to $\lambda = 0$, are

$$\begin{cases} u_r = \cos(\theta) \\ u_\theta = -\sin(\theta) \end{cases} \tag{4.1a}$$

$$\begin{cases} u_r = \sin(\theta) \\ u_\theta = \cos(\theta) \end{cases} \tag{4.1b}$$

which are pure translation modes in x_1 and x_2 directions respectively, and

$$\begin{cases} u_r = 0 \\ u_\theta = r \end{cases} \tag{4.1c}$$

which is the pure rotation mode.

We can show that one set of non-rigid body fundamental eigensolutions corresponding to $\varphi_{ij} = \delta_{ij}$ are

$$\begin{cases} u_r^1 = r^n \cos[(n+1)\theta] \\ u_\theta^1 = -r^n \sin[(n+1)\theta] \end{cases} \tag{4.2a}$$

and

$$\begin{cases} u_r^2 = r^n \sin[(n+1)\theta] \\ u_\theta^2 = r^n \cos[(n+1)\theta] \end{cases} \tag{4.2b}$$

with eigenvalues

$$\lambda = \frac{2\tilde{\mu}n}{a} \tag{4.3}$$

where $n = 1, 2, \ldots$ These modes, which we designate as type I, are equivoluminal deformations. It can be seen that the translational rigid body eigenmodes (4.1a) and (4.1b) can be categorized as type I if we put $n = 0$ in Eqs. (4.2a), (4.2b) and (4.3).

The other eigenmodes, which we call type II, are

$$\begin{cases} u_r^3 = [\kappa r^n - n(r^2 - a^2)r^{n-2}]\cos[(n-1)\theta] \\ u_\theta^3 = [\kappa r^n + n(r^2 - a^2)r^{n-2}]\sin[(n-1)\theta] \end{cases} \tag{4.4a}$$

$$\begin{cases} u_r^4 = [-\kappa r^n + n(r^2 - a^2)r^{n-2}]\sin[(n-1)\theta] \\ u_\theta^4 = [\kappa r^n + n(r^2 - a^2)r^{n-2}]\cos[(n-1)\theta] \end{cases} \tag{4.4b}$$

corresponding to

$$\lambda = \frac{2\tilde{\mu} n}{\kappa a} \tag{4.5}$$

where $\kappa = 3 - 4v$ and $n = 2, 3, \ldots$

The stresses corresponding to type I and II eigenmodes are

$$\begin{cases} \sigma_r^1 = 2\tilde{\mu} n r^{n-1}\cos[(n+1)\theta] \\ \tau_{r\theta}^1 = -2\tilde{\mu} n r^{n-1}\sin[(n+1)\theta] \\ \sigma_\theta^1 = -2\tilde{\mu} n r^{n-1}\cos[(n+1)\theta] \end{cases} \tag{4.6a}$$

$$\begin{cases} \sigma_r^2 = 2\tilde{\mu} n r^{n-1}\sin[(n+1)\theta] \\ \tau_{r\theta}^2 = 2\tilde{\mu} n r^{n-1}\cos[(n+1)\theta] \\ \sigma_\theta^2 = -2\tilde{\mu} n r^{n-1}\sin[(n+1)\theta] \end{cases} \tag{4.6b}$$

$$\begin{cases} \sigma_r^3 = 2\tilde{\mu} n[r^{n-1} - (n-2)(r^2 - a^2)r^{n-3}]\cos[(n-1)\theta] \\ \tau_{r\theta}^3 = 2\tilde{\mu} n[r^{n-1} + (n-2)(r^2 - a^2)r^{n-3}]\sin[(n-1)\theta] \\ \sigma_\theta^3 = 2\tilde{\mu} n[3r^{n-1} + (n-2)(r^2 - a^2)r^{n-3}]\cos[(n+1)\theta] \end{cases} \tag{4.6c}$$

$$\begin{cases} \sigma_r^4 = 2\tilde{\mu} n[-r^{n-1} + (n-2)(r^2 - a^2)r^{n-3}]\sin[(n-1)\theta] \\ \tau_{r\theta}^4 = 2\tilde{\mu} n[r^{n-1} + (n-2)(r^2 - a^2)r^{n-3}]\cos[(n-1)\theta] \\ \sigma_\theta^4 = -2\tilde{\mu} n[3r^{n-1} + (n-2)(r^2 - a^2)r^{n-3}]\sin[(n-1)\theta] \end{cases} \tag{4.6d}$$

There is also one single eigenmode (type III)

$$\begin{cases} u_r = r \\ u_\theta = 0 \end{cases} \tag{4.7}$$

with

$$\lambda = \frac{4\tilde{\mu}}{(\kappa - 1)a} = \frac{2\tilde{\mu}}{(1 - 2v)a} \tag{4.8}$$

The stress corresponding to this eigenmode is

$$\begin{cases} \sigma_r = \sigma_\theta = \frac{2\tilde{\mu}}{1-2v} \\ \tau_{r\theta} = 0 \end{cases} \tag{4.9}$$

This deformation is completely radial and dilatational.

It should be noted that the fundamental boundary condition here is

$$\begin{cases} t_r = \sigma_r = \lambda u_r & \text{when } r = a \\ t_\theta = \tau_{r\theta} = \lambda u_\theta & \text{when } r = a \end{cases} \tag{4.10}$$

and the orthogonality of eigenmodes in polar coordinates becomes

$$\int_0^{2\pi} [u_r^{(m)}(a,\theta)u_r^{(n)}(a,\theta) + u_\theta^{(m)}(a,\theta)u_\theta^{(n)}(a,\theta)]a\,\mathrm{d}\theta = 0, \quad m \neq n \tag{4.11}$$

Furthermore, it is obvious that all of the non-zero eigenvalues, except the one corresponding to the type III radial eigenmode, are repeated. One interesting feature common among eigenmodes is that the stresses in Eqs. (4.6a)–(4.6d) and (4.9) are independent of ν (or κ), except perhaps for a constant normalization factor. This finding is in agreement with the well-known independence of the stress distribution from the Poisson ratio in a 2-D simply connected body.

Next we examine the boundary eigensolutions obtained numerically for the plane strain deformation of the circle. Let the radius $a = 1$ and set the material properties $E = 1.0$ and $\nu = 0.3$. We use a boundary element method (BEM) based upon the integral equation (3.15). Additional details concerning the numerical implementation can be found in Hadjesfandiari (1998) and in Part II. The discretized model uses 36 quadratic elements for both geometric and functional variation. A partial list of eigenvalues are given in Table 1 along with exact values. The deformations for modes 8, 14, 18, 25, 27 and 38 are shown in Figs. 1–6.

The 14th eigenmode corresponds to radial deformations, for which we have the closed form eigenvalue

$$\lambda = \frac{2\tilde{\mu}}{a(1-2\nu)} = \frac{E}{a(1-2\nu)(1+\nu)}$$

By substituting numerical values we obtain

$$\lambda = 1.92307$$

Table 1
Boundary eigenvalues for unit circular disc ($E = 1$, $\nu = 0.3$, $a = 1$)

Mode	Type	Exact	BEM
1	RB	0.00000	0.43921×10^{-16}
2	RB	0.00000	0.18575×10^{-15}
3	RB	0.00000	0.69423×10^{-15}
5	I	0.76923	0.76923
8	II	1.2821	1.2823
11	I	1.5385	1.5385
14	III	1.9231	1.9231
18	I	2.3077	2.3077
20	II	2.5641	2.5697
25	II	3.4188	3.4394
27	I	3.8462	3.8465
32	II	4.2735	4.3301
35	II	4.7701	4.7879
38	II	5.1282	5.2570
44	I	6.1538	6.1580
50	I	6.9231	6.9291
55	II	7.6923	7.8640
60	I	8.4615	8.4611
65	II	9.4017	9.6650
70	I	10.000	10.177
75	II	11.966	12.482
80	I	13.077	13.105

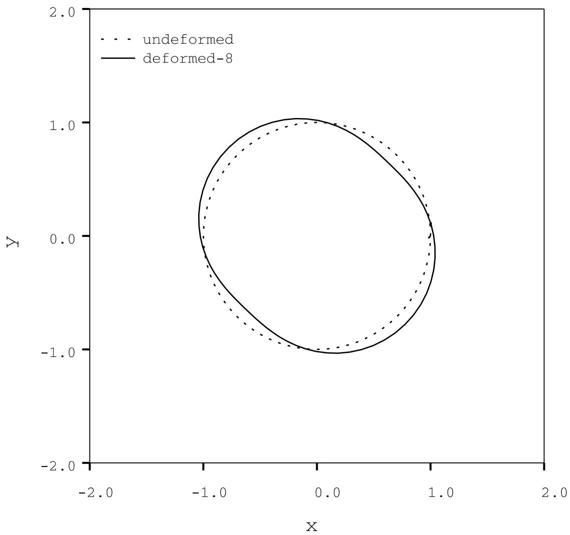

Fig. 1. Unit circular disc – eigenmode for mode 8.

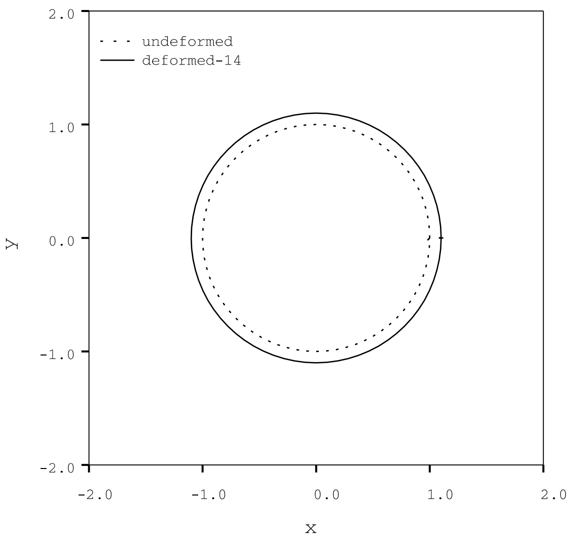

Fig. 2. Unit circular disc – eigenmode for mode 14.

This value is nearly the same as the numerical value from BEM as listed in Table 1. This radial eigenmode shows the deformation of the body under axisymmetric loading. The normalized form of this eigenmode is

$$u_r = \frac{1}{\sqrt{2\pi a}}\frac{r}{a}$$

$$u_\theta = 0$$

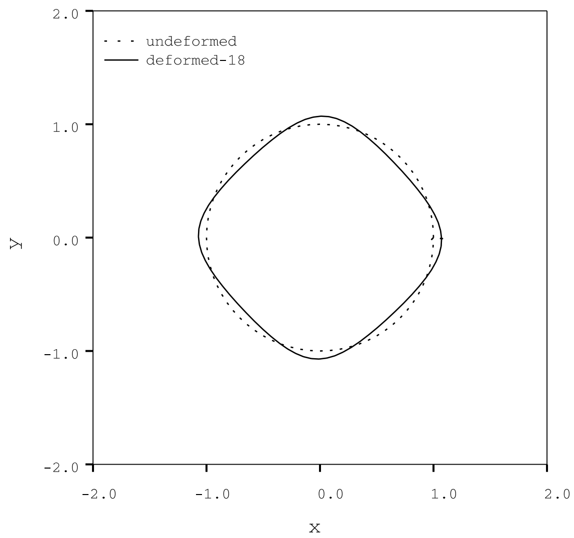

Fig. 3. Unit circular disc – eigenmode for mode 18.

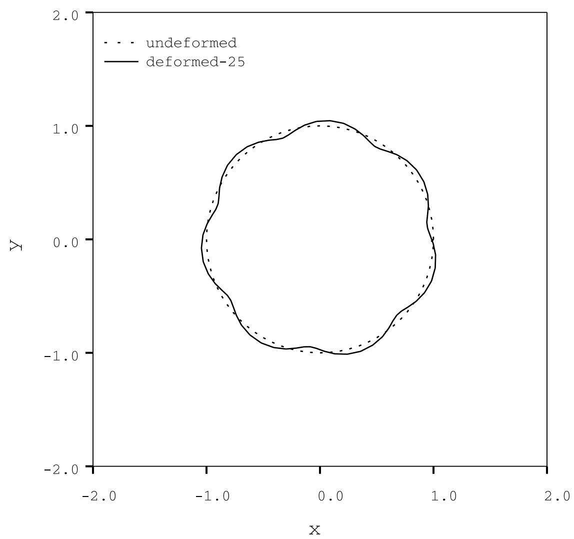

Fig. 4. Unit circular disc – eigenmode for mode 25.

Modes 18 and 27 are shear deformations for which we derived the closed form expressions (4.2a) and (4.2b). The normalized eigenmode 18 corresponding to $\lambda_{18} = (6\tilde{\mu}/a) = 2.3077$ can be associated with either

$$u_r = \frac{1}{\sqrt{2\pi a}}\left(\frac{r}{a}\right)^3\cos(4\theta) \quad \text{or} \quad \frac{1}{\sqrt{2\pi a}}\left(\frac{r}{a}\right)^3\sin(4\theta)$$
$$u_\theta = -\frac{1}{\sqrt{2\pi a}}\left(\frac{r}{a}\right)^3\sin(4\theta) \quad \text{or} \quad \frac{1}{\sqrt{2\pi a}}\left(\frac{r}{a}\right)^3\cos(4\theta)$$

and eigenmode 27 corresponding to $\lambda_{27} = (10\tilde{\mu}/a) = 3.8462$ can be associated with either

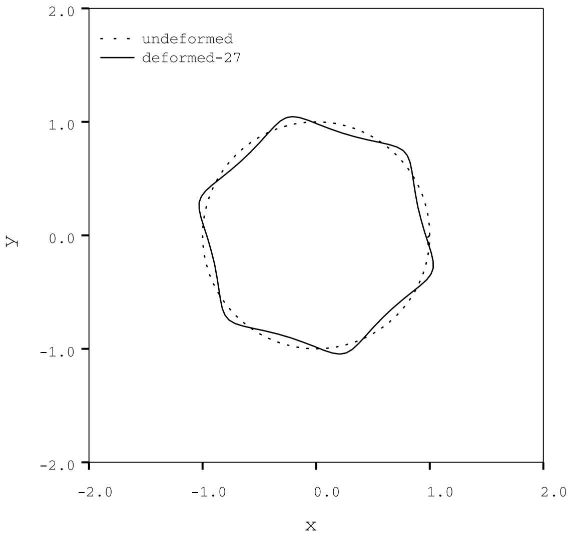

Fig. 5. Unit circular disc – eigenmode for mode 27.

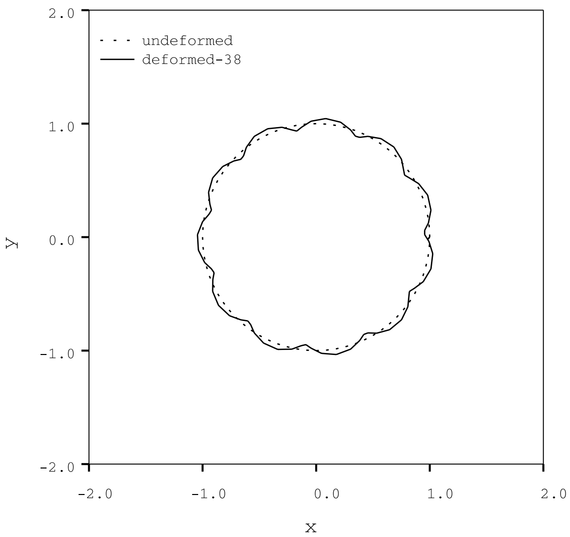

Fig. 6. Unit circular disc – eigenmode for mode 38.

$$u_r = \frac{1}{\sqrt{2\pi a}}\left(\frac{r}{a}\right)^5 \cos(6\theta) \quad \text{or} \quad \frac{1}{\sqrt{2\pi a}}\left(\frac{r}{a}\right)^5 \sin(6\theta)$$
$$u_\theta = -\frac{1}{\sqrt{2\pi a}}\left(\frac{r}{a}\right)^5 \sin(6\theta) \quad \text{or} \quad \frac{1}{\sqrt{2\pi a}}\left(\frac{r}{a}\right)^5 \cos(6\theta)$$

As we can see from Table 1, the eigensolutions derived from BEM for higher modes are less accurate. However, the eigenvalues improve by mesh refinement and the eigenmodes approach to the exact shapes. In general, we can also observe that the BEM generates more accurate equivoluminal eigensolutions (type I).

5. Elastostatic problem with body force

All of the above theory pertains to the solution of the Navier equation in the absence of body force. We now examine the case in which body forces $\boldsymbol{b}(x)$ are present in the volume. Thus, for equilibrium

$$\sigma_{ij,j} + b_i = 0 \tag{5.1}$$

$$\sigma_{ij} = \sigma_{ji} \tag{5.2}$$

By substituting from Eq. (2.4) in Eq. (5.1), we derive the Navier equations as

$$C_{ijkl} u_{k,lj} + b_i = 0 \tag{5.3}$$

with the following conditions specified on the boundary

$$\boldsymbol{u} = \boldsymbol{f}(x) \quad \text{on } S_u \tag{5.4a}$$

$$\boldsymbol{t} = \boldsymbol{g}(x) \quad \text{on } S_t \tag{5.4b}$$

We know, however, that the solution can be expressed as the combination of two solutions

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{I}} + \boldsymbol{u}^{\mathrm{II}} \tag{5.5}$$

where $\boldsymbol{u}^{\mathrm{I}}$ and $\boldsymbol{u}^{\mathrm{II}}$ represent homogeneous and particular solutions, respectively. The homogeneous solution is the solution of

$$C_{ijkl} u^{\mathrm{I}}_{k,lj} = 0 \tag{5.6}$$

such that on the boundary

$$\boldsymbol{u}^{\mathrm{I}} = \boldsymbol{f}(x) \quad \text{on } S_u \tag{5.7a}$$

$$\boldsymbol{t}^{\mathrm{I}} = \boldsymbol{g}(x) \quad \text{on } S_t \tag{5.7b}$$

This solution follows the theory of generalized fundamental boundary eigensolutions. Consequently,

$$\boldsymbol{u}^{\mathrm{I}} = \sum_{n=1}^{\infty} A_n \boldsymbol{u}^{(n)} \quad \text{in } V \cup S \tag{5.8a}$$

$$\boldsymbol{t}^{\mathrm{I}} = \Phi \cdot \sum_{n=1}^{\infty} A_n \lambda_n \boldsymbol{u}^{(n)} \quad \text{on } S \tag{5.8b}$$

On the other hand, $\boldsymbol{u}^{\mathrm{II}}$ is then the particular solution of

$$C_{ijkl} u^{\mathrm{II}}_{k,lj} + b_i = 0 \tag{5.9}$$

under the homogeneous boundary conditions

$$\boldsymbol{u}^{\mathrm{II}} = \boldsymbol{0} \quad \text{on } S_u \tag{5.10a}$$

$$\boldsymbol{t}^{\mathrm{II}} = \boldsymbol{0} \quad \text{on } S_t \tag{5.10b}$$

This latter solution follows the traditional theory of eigensolutions of the vibration equation. This eigenproblem can be written

$$C_{ijkl} v_{k,lj} = \omega^2 \rho v_i \tag{5.11}$$

with homogeneous boundary conditions

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } S_u \tag{5.12a}$$

$$\boldsymbol{t} = \mathbf{0} \quad \text{on } S_t \tag{5.12b}$$

where $\rho(x)$ is an integrable positive weight function in the domain V and ω^2 is the eigenvalue.

Details concerning the properties of the resulting eigensolutions (ω_n^2, v_n) can be found in Courant and Hilbert (1953) and Eringen and Suhubi (1974). These eigensolutions have familiar properties. For example, the eigenvalues are real and the sequence of eigenfunctions are orthogonal with respect to ρ in the domain V

$$\int_V \rho v_m v_n \, \mathrm{d}V = 0 \quad \text{for } m \neq n$$

It should be emphasized that these eigensolutions are not orthogonal on the boundary S in general. By normalizing eigensolutions we have

$$\int_V \rho v_m^2 \, \mathrm{d}V = 1, \quad m = 1, 2, \ldots$$

One result is that the deformation $\boldsymbol{u}^{\mathrm{II}}$ may be expressed in terms of a series of these eigensolutions. Thus, with $\rho = 1$, we have

$$\boldsymbol{u}^{\mathrm{II}} = \sum_{n=1}^{\infty} B_n \boldsymbol{v}^{(n)} \quad \text{in } V \cup S \tag{5.13a}$$

where

$$B_n = -\frac{1}{\omega^2} \int_V \boldsymbol{b} \cdot \boldsymbol{v}^{(n)} \, \mathrm{d}V \tag{5.13b}$$

Substituting Eqs. (5.8a) and (5.13a) into Eq. (5.5), we find that the solution of Eq. (5.3) with boundary conditions (5.4a) and (5.4b) can be written

$$\boldsymbol{u} = \sum_{n=1}^{\infty} A_n \boldsymbol{u}^{(n)} + \sum_{n=1}^{\infty} B_n \boldsymbol{v}^{(n)} \quad \text{in } V \cup S \tag{5.14}$$

It is interesting that the solution appears as the combination of two different series of orthogonal deformations. One series is orthogonal on the boundary, while the other is orthogonal over the domain.

6. Concluding remarks

In this paper, we have explored the concept of boundary eigensolutions to elastostatic boundary value problems. The energetic character of these eigensolutions have an amazing impact in the theory of elasticity. The resulting theory furnishes new insight into the solutions of BVPs. In addition, we find that there is a connection among the theory of boundary eigensolutions, integral equation methods and variational methods. In the domain of computational mechanics, this provides a relationship between BEMs and finite element methods. The flexibility and stiffness tensors show the deep influence of fundamental boundary eigensolutions in the theory of elasticity.

Hilbert (1912) has mentioned the boundary eigensolutions with $\varphi = 1$ long ago for the potential problem, and has even given their relation with the calculus of variations. He did not notice the relation with the direct integral equation. This theory has been further developed here for elastostatics by introducing a general positive definite weight tensor function φ_{ij}, which then provides a unified treatment for non-smooth problems in engineering mechanics and allows for meaningful solutions to be obtained. The analytic eigensolutions corresponding to $\varphi_{ij} = \delta_{ij}$ are the natural result of the traditional integral equation, but are not powerful for the analysis of non-smooth problems.

The simple numerical example considered in Section 4, based on a boundary element formulation, illustrates some aspects of this new methodology. Convergence in the mean, uniformity of the solution and Gibbs' phenomenon are the tools for use in approximation methods, including computational mechanics, which will be discussed in detail in Part II.

Appendix A. Orthonormal rigid body eigenmodes

The number of rigid body modes is $n_{\mathrm{R}} = 6$ in three dimensions and $n_{\mathrm{R}} = 3$ in two dimensions. We derive the orthonormalized rigid body eigenmodes corresponding to $\lambda = 0$ with positive definite $\varphi_{ij} = \varphi\delta_{ij}$. For this case, the normalizing relation is simply

$$\int_S \varphi u_i u_i \, \mathrm{d}S = 1$$

Let S be the magnitude of the length or the area of the boundary and assume φ as a surface mass density. We define the first mass moment and second mass moment (inertia tensor) as

$$M_i = \int_S \varphi x_i \, \mathrm{d}S$$

$$I_{ij} = \int_S \varphi (x_k x_k \delta_{ij} - x_i x_j) \, \mathrm{d}S$$

For simplicity, we assume the origin is at the center of the mass of the boundary such that

$$M_i = 0$$

and the coordinate axes are principal axes of the boundary surface such that the inertia tensor is diagonal

$$\int_S \varphi x_1 x_2 \, \mathrm{d}S = \int_S \varphi x_2 x_3 \, \mathrm{d}S = \int_S \varphi x_3 x_1 \, \mathrm{d}S = 0$$

Let the principal inertia be

$$I_1 = \int_S \varphi (x_2^2 + x_3^2) \, \mathrm{d}S$$
$$I_2 = \int_S \varphi (x_3^2 + x_1^2) \, \mathrm{d}S$$
$$I_3 = \int_S \varphi (x_1^2 + x_2^2) \, \mathrm{d}S$$

It is easily seen that the three translational eigenmodes are

$$u_i^{(1)} = \frac{1}{\sqrt{\int_S \varphi \, \mathrm{d}S}} \delta_{1i}$$

$$u_i^{(2)} = \frac{1}{\sqrt{\int_S \varphi \, \mathrm{d}S}} \delta_{2i}$$

$$u_i^{(3)} = \frac{1}{\sqrt{\int_S \varphi \, \mathrm{d}S}} \delta_{3i}$$

or in general

$$u_i^{(r)} = \frac{1}{\sqrt{\int_S \varphi\, dS}} \delta_{ri}$$

which are translations parallel to principal axes. The normalized translation amplitude is $1/(\int_S \varphi\, dS)^{1/2}$. Also we can see that the three rotational modes are

$$u_i^{(4)} = \frac{1}{\sqrt{I_1}} \varepsilon_{i1k} x_k$$
$$u_i^{(5)} = \frac{1}{\sqrt{I_2}} \varepsilon_{i2k} x_k$$
$$u_i^{(6)} = \frac{1}{\sqrt{I_3}} \varepsilon_{i3k} x_k$$

which are rotations about principal axes. If the moment of inertia of the boundary surface about an axis p is I_p, then the magnitude of rotation is $1/(I_p)^{1/2}$.

In the 2-D case, the rigid body eigenmodes are

$$u_i^{(1)} = \frac{1}{\sqrt{\int_S \varphi\, dS}} \delta_{1i}$$
$$u_i^{(2)} = \frac{1}{\sqrt{\int_S \varphi\, dS}} \delta_{2i}$$
$$u_i^{(3)} = \frac{1}{\sqrt{I_3}} \varepsilon_{i3k} x_k$$

It is seen that these rigid body eigenmodes are orthogonal to each other.

Furthermore, for the 3-D case we have

$$\sum_{m=1}^{3} u_i^{(m)}(x) u_j^{(m)}(y) = \frac{1}{\int_S \varphi\, dS} \delta_{ij}$$

and

$$\left[\sum_{m=4}^{6} u_i^{(m)}(x) u_j^{(m)}(y)\right] = \begin{bmatrix} \frac{1}{I_2} x_3 y_3 + \frac{1}{I_3} x_2 y_2 & -\frac{1}{I_3} x_1 y_2 & -\frac{1}{I_2} x_1 y_3 \\ -\frac{1}{I_3} x_2 y_1 & \frac{1}{I_1} x_3 y_3 + \frac{1}{I_3} x_1 y_1 & -\frac{1}{I_1} x_2 y_3 \\ -\frac{1}{I_2} x_3 y_1 & -\frac{1}{I_1} x_3 y_2 & \frac{1}{I_1} x_2 y_2 + \frac{1}{I_2} x_1 y_1 \end{bmatrix}$$

while for the 2-D case

$$\sum_{m=1}^{2} u_i^{(m)}(x) u_j^{(m)}(y) = \frac{1}{\int_S \varphi\, dS} \delta_{ij}$$

and

$$[u_i^{(3)}(x) u_j^{(3)}(y)] = \frac{1}{I_3} \begin{bmatrix} x_2 y_2 & -x_1 y_2 \\ -x_2 y_1 & x_1 y_1 \end{bmatrix}$$

Appendix B. Principal mean value

Suppose a piecewise regular function $f(x)$ is defined on the boundary S.

B.1. Two dimensional domains

Let L_t represent the length of arc inscribed in the circle with radius r and centered at ξ on S. The *principal mean value* of $f(x)$ at ξ is defined as

$$\hat{f}(\xi) = \lim_{r\to 0}\frac{1}{L_t}\int_{L_t} f(x)\,\mathrm{d}S(x) \tag{B.1}$$

If we define the limits of $f(x)$ from both directions at ξ by $f^{\mathrm{I}}(\xi)$ and $f^{\mathrm{II}}(\xi)$, then the principal mean value at ξ is

$$\hat{f}(\xi) = \tfrac{1}{2}[f^{\mathrm{I}}(\xi) + f^{\mathrm{II}}(\xi)] \tag{B.2}$$

If $f(x)$ is continuous at ξ, the principal mean value is $f(\xi)$ itself.

B.2. Three dimensional domains

Let A_t represent the area of surface inscribed in the sphere with radius r and centered at ξ on S. The *principal mean value* of $f(x)$ at ξ is defined as

$$\hat{f}(\xi) = \lim_{r\to 0}\frac{1}{A_t}\int_{A_t} f(x)\,\mathrm{d}S(x) \tag{B.3}$$

If $f(x)$ is continuous at ξ, the principal mean value is $f(\xi)$ itself.

Appendix C. Boundary Dirac-delta function

We define the Dirac-delta function $\delta_S(x,\xi)$ on the boundary S such that

$$\delta_S(x,\xi) = 0, \quad x \neq \xi \tag{C.1}$$

and

$$\int_S \delta_S(x,\xi)\,\mathrm{d}S(x) = 1, \quad \xi \in S \tag{C.2}$$

and for every L_2-function $f(x)$

$$\int_S \delta_S(x,\xi) f(x)\,\mathrm{d}S(x) = \hat{f}(\xi), \quad \xi \in S \tag{C.3}$$

where $\hat{f}(\xi)$ is the principal mean value of function $f(x)$ at ξ (Appendix B).

References

Banerjee, P.K., 1994. The Boundary Element Methods in Engineering. McGraw-Hill, London.
Carslaw, H.S., 1950. An Introduction to the Theory of Fourier's Series and Integrals. Dover, New York.
Courant, R., Hilbert, D., 1953. Methods of Mathematical Physics. Wiley, New York.
Eringen, A.C., Suhubi, E.S., 1974. Elastodynamics. Academic Press, New York.

Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. Roy. Soc. A 241, 376–396.

Garabedian, P.R., 1988. Partial Differential Equations. AMS Chelsea Publishing, Rhode Island.

Hadjesfandiari, A.R., 1998. Theoretical and computational concepts in engineering mechanics. Ph.D. dissertation, State University of New York at Buffalo, Buffalo.

Hadjesfandiari, A.R., Dargush, G.F., 2001a. Theory of boundary eigensolutions in engineering mechanics. J. Appl. Mech., ASME 68, 101–108.

Hadjesfandiari, A.R., Dargush, G.F., 2001b. Computational mechanics based on the theory of boundary eigensolutions. Int. J. Numer. Meth. Eng. 50, 325–346.

Hadjesfandiari, A.R., Dargush, G.F., 2001c. Boundary eigensolutions in elasticity. II. Application in computational mechanics. Int. J. Solids Struct., submitted for publication.

Hilbert, D., 1912. Grundzuge einer Allgemeinen Theorie der Linearen Integralgleichungen, Teubner, Leipzig.

Kanwal, P.K., 1971. Linear Integral Equations. Academic Press, New York.

Kupradze, V.D., 1979. Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity. North Holland, Amsterdam.

Muskhelishvili, N.I., 1953. Some Basic Problems of the Mathematical Theory of Elasticity, Noordhoff, Groningen, Holland.

Noble, B., 1958. Methods Based on the Wiener–Hopf Technique for the Solution of Partial Differential Equations. Pergamon, New York.

Parton, V.Z., Perlin P.I., 1977. Integral Equations in Elasticity. Mir Publishers, Moscow.

Sneddon, I.N., 1966. Mixed Boundary Value Problems in Potential Theory. North Holland, Amsterdam.

Williams, M.L., 1952. Stress singularities resulting from various boundary conditions in angular corners of plates in extension. J. Appl. Mech., ASME 19, 526–528.